GLOBAL SOLVABILITY OF A NETWORKED INTEGRATE-AND-FIRE MODEL OF MCKEAN-VLASOV TYPE

F. DELARUE, J. INGLIS, S. RUBENTHALER, E. TANRÉ

Abstract. We here investigate the well-posedness of a networked integrate-and-fire model describing an infinite population of neurons which interact with one another through their common statistical distribution. The interaction is of the self-excitatory type as, at any time, the potential of a neuron increases when some of the others fire: precisely, the kick it receives is proportional to the instantaneous proportion of firing neurons at the same time. From a mathematical point of view, the coefficient of proportionality, denoted by α, is of great importance as the resulting system is known to blow-up for large values of α. In the current paper, we focus on the complementary regime and prove that existence and uniqueness hold for all time when α is small enough. The critical value for existence and uniqueness is made explicit in some popular examples found in the neuroscience literature and is also compared with numerical experiments.

Keywords: McKean nonlinear diffusion process; renewal process; density estimates; integrate-and-fire model; network of neurons; neuroscience.

1. Introduction

The stochastic integrate-and-fire model of the membrane potential V across a neuron in the brain has received a huge amount of attention since its introduction (see [18] for a comprehensive review). The central idea is to model V by threshold dynamics, in which the potential is described by a simple linear (stochastic) differential equation up until it reaches a fixed threshold value V_F, at which point the neuron emits a 'spike'. Experimentally, at this point an action potential is observed, whereby the potential increases very rapidly to a peak (hyperpolarization phase) before decreasing quickly to a reset value (depolarization phase). It then relatively slowly increases once more to the resting potential (refractory period).

Since spikes are stereotyped events, they are fully characterized by the times at which they occur. The integrate-and-fire model is part of a family of spiking neuron models which take advantage of this by modeling only the spiking times and disregarding the nature of the spike itself. Specifically, in the integrate-and-fire model we observe jumps in the action potential as the voltage is immediately reset to a value V_R whenever it reaches the threshold V_F, which is motivated by the fact that the time period during which the action potential is observed is very small. Despite its simplicity, the integrate-and-fire model has been able to predict the spiking times

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of a neuron with a reasonable degree of accuracy [10, 11], and as such provides the basis for a good first study.

Many extensions of the basic integrate-and-fire model have been studied, including ones in which attempts are made to include noise and to describe the situation when many integrate-and-fire neurons are placed in a network and interact with each other. Indeed, in [13] and [14] the following equation describing how the potential V_i of the i-th neuron in a network of N behaves in time is proposed:

$$\frac{d}{dt}V_i(t) = -\lambda V_i(t) + \frac{\alpha}{N}\sum_j \sum_{n_j=-\infty}^{\infty} \delta(t - t_{n_j}) + \frac{\beta}{N}\sum_{j\neq i} V_j(t) + I_i^{ext}(t) + \sigma\eta_i(t) \quad (1.1)$$

for $V_i(t) < V_F$ and where $V_i(t)$ is immediately reset to V_R when it reaches V_F. Here $I_i^{ext}(t)$ represents the external input current to the neuron, $\eta_i(t)$ is the noise (a white noise) which is importantly supposed to be independent from neuron to neuron, and the constants λ, β, α and σ are chosen according to experimental data. Moreover, the interaction term is described in terms of t_{n_j}, which is the time of the n-th spike of neuron j, and the Dirac function δ. Precisely, it says that whenever one of the other neurons in the network spikes, the potential across neuron i receives a 'kick' of size α/N.

In the case of a large network i.e. when N is large, many authors approximate the interaction term in the above finite system by an instantaneous rate $\nu(t)$, the so-called mean-firing rate (see for example [1, 2, 14, 16]). However, in the neuroscience literature little attention is paid to how this convergence is achieved. Mathematically the mean-field limit as $N \to \infty$ must be taken, and it is not at all clear that one observes a propagation of chaos phenomenon. This is the subject of a forthcoming manuscript [5].

Just as important is the question of whether solutions exist for the resulting non-linear equation when the mean-field limit is taken, and if so what type of solutions are admissible. Again this is not treated rigorously in the neuroscience literature, and it is in this diction that we proceed here. Since the mathematical difficulties lie within the jump interaction term, we in fact isolate it by making two simplifications. Indeed, we suppose that there is no external input current ($I_i^{ext}(t) \equiv 0$), and that the interaction term is composed solely of the jump or reset part ($\beta = 0$). Although this is a big simplification from a neuroscience perspective, it still captures all the mathematical complexity of the resulting mean-field equation, and standard approaches (see for example [20]) would then allow extensions to more realistic models.

Without loss of generality, we also take the firing threshold $V_F = 1$ and the reset value $V_R = 0$ for notational simplicity. We finally choose to describe the problem in the language of stochastic processes, so that the precise nonlinear stochastic differential equation under study here becomes

$$X_t = X_0 + \int_0^t b(X_s)ds + \alpha\mathbb{E}(M_t) + \sigma W_t - M_t, \qquad t \geqslant 0, \quad (1.2)$$

where $X_0 < 1$ almost surely, $\alpha \in \mathbb{R}$, $\sigma > 0$, W_t is a standard Brownian motion in $\mathbb{R}$ and $b : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous. The jumps, or resets, of the system are

described by

$$M_t = \sum_{k \geqslant 1} \mathbf{1}_{[0,t]}(\tau_k)$$

where $(\tau_k)_{k \geqslant 1}$ stands for the sequence of hitting times of 1 by the process $(X_t)_{t \geqslant 0}$. That is $(M_t)_{t \geqslant 0}$ counts the number of times X_t hits the threshold before time t, so that $\mathbb{E}(M_t)$ denotes the expected number of times the threshold is reached before t. This equation corresponds to what we would expect when the limit $N \to \infty$ is taken in (1.1) when the above simplifications are made, where the resets are now included in the equation by the presence of the counting process $(M_t)_{t \geqslant 0}$. It is indeed the case that the associated finite system of N neurons converges to this nonlinear limit equation (again this is shown in our forthcoming manuscript [5]).

In fact a less general version of equation (1.2) has been rigorously studied from the PDE viewpoint before, though not completely satisfactorily. If one applies the generalized Itô formula for jump processes to $f(X_t)$ for a suitable function f, then in the standard way one arrives at the Fokker-Planck equation for the density $p(t,y)dy = \mathbb{P}(X_t \in dy)$:

$$\partial_t p(t,y) + \partial_y \left[\left(b(y) + \alpha e'(t) \right) p(t,y) \right] - \frac{1}{2}\partial^2_{yy} p(t,x) = \delta_0(y - V_R) e'(t), \qquad y < 1,$$

where $e(t) = \mathbb{E}(M_t)$, subject to $p(t,1) = 0$, $p(t,-\infty) = 0$, $p(0,y)dy = \mathbb{P}(X_0 \in dy)$ and where δ_0 is the Dirac measure at 0. Moreover, the condition that $p(t,y)$ must remain a probability density translates into the fact that we must have

$$e'(t) = \frac{d}{dt}\mathbb{E}(M_t) = -\frac{1}{2}\partial_y p(1,t), \qquad \forall t > 0.$$

Now, in the case when $b(x) = -x$, this nonlinear Fokker-Planck equation is exactly the one studied in [3] and [4]. Therein, the authors define a solution to be one in which $e'(t)$ remains bounded, since the explosion of $e'(t)$ corresponds to the situation where a large proportion of the neurons in the network all spike at the same time. Given this definition, the equation is shown to be uniquely solvable in the cases $\alpha = 0$ and $\alpha < 0$ (see Theorem 1.1 in [4]), the latter one being referred to as 'self-inhibitory' in neuroscience. In the so-called 'self-excitatory' framework, i.e. for $\alpha > 0$, existence of a solution for all time has been left open. Instead, very interestingly, a negative result was established in [3], stating that, for any $\alpha > 0$, it is possible to find an initial probability distribution $\mathbb{P}(X_0 \in dy)$ such that there is a blow-up, i.e. such that $e'(t) = \infty$ in finite time. As a remarkable point, it says that solvability in the long run may fail for possibly small values of α. On the contrary, it does not come as a surprise that solutions may explode in a finite time for the prescribed values of V_R and V_F when $\alpha \geqslant 1$: indeed, if Np particles in (1.1) reach the threshold $V_F = 1$ between some times t_1 and t_2 for some proportion $p \in (0,1]$, then all the particles perform a jump of size greater than $\alpha p(N-1)/N$ within the same period, which reads as a size greater than αp in the ideal case $N = \infty$ corresponding to (1.2); in particular, when $\alpha p \geqslant 1$ and the reset is fixed as 0, particles keep on crossing the threshold an infinite number of times between t_1 and t_2. Such a picture

indicates that the system enters a sort of 'super-self-excitatory' regime when α is above $V_F - V_R$, which matches 1 with our convention.

Here we thus investigate the case $\alpha \in (0,1)$. We then contribute the associated positive and until now unknown result, by proving that, given a starting point $X_0 = x_0$, we can find an explicit α small enough so that there is an unique solution to (1.2) (and hence to the associated Fokker-Planck equation) for all time, such that $e'(t) = [d/dt]\mathbb{E}(M_t)$ remains bounded on any compact subset (see Theorem 5.2). The surprising difficulty of this result is reflected in the rather involved nature of our proofs.

Our result is for a general Lipschitz function b, but there are two important specific cases that we study further: the Brownian case when $b \equiv 0$ and the Ornstein-Uhlenbeck case when $b(x) = -\lambda x$, $\lambda \geqslant 0$. The Ornstein-Uhlenbeck case is most relevant to neuroscience since it is often used in the literature, but surprising difficulties remain in the purely Brownian case. In both these cases we are able to give an explicit α_0 depending on the starting point x_0 such that (1.2) has a solution for all $\alpha < \alpha_0$. However, it is interesting to note that our explicit values do not appear to be optimal, in the sense that from our numerical simulations there appears to be a real 'gap' between our positive result and the negative result of [3]. Precisely, our simulations show that for a given x_0 there exist classical solutions for all time for α bigger than our explicit α_0, while there exist solutions that blow-up that do not satisfy the conditions of [3]. Thus a very interesting, and perhaps very difficult question is to determine for a given initial starting point the true critical value α_c such that for $\alpha < \alpha_c$ (1.2) has an unique global solution. We discuss this further at the end of the paper.

Equation (1.2) can be thought as of McKean-Vlasov-type, since the process X_t depends on the distribution of the solution itself. However, it is highly non-standard, since it actually depends on the distribution of the *hitting times* of the threshold by the solution. This renders the traditional approaches to McKean-Vlasov equations and propagation of chaos, such as those presented by Sznitman in [20], inapplicable, because we have no *a priori* smoothness on the law of the hitting times. Thus our results are also new in this context.

The general structure of the proof is quite typical of the methods used to investigate well-posedness of nonlinear systems, and especially well-posedness of Markovian stochastic differential equations (or equivalently of partial differential equations of second order) involving some non-trivial nonlinearity. Precisely, the first point is to tackle unique solvability in small time: when the parameter α is (strictly) less than 1 and the mass of the initial condition decays linearly at the threshold, it is proven that the system induces a natural contraction in a well-chosen space provided the time duration is small enough. In this framework, the specific notion of solution at hand plays a crucial role as it defines the right space for the contraction. Below, solutions are sought in such a way that the mapping $t \mapsto \mathbb{E}(M_t)$ is continuously differentiable. The proof is at the intersection between probability and PDEs. The second stage is then to extend existence and uniqueness from short to long times. Exactly as for PDEs of quasilinear type, the point is to prove that some key quantity

is preserved as time goes by. Here, we prove that the system cannot accumulate too much mass in the right vicinity of 1. Equivalently, this amounts to showing that the Lipschitz constant of the mapping $t \mapsto \mathbb{E}(M_t)$ cannot blow-up in a finite time. This is where the condition α small enough comes in: when α is small enough, we manage to give some estimates for the density of X_t in the neighbourhood of 1, the critical value of α explicitly depending upon the available bound of the density. Generally speaking, we make use of standard Gaussian estimates of Aronson type of the density; in the cases $b \equiv 0$ and $b(x) = -\lambda x$, $x \in \mathbb{R}$, for some $\lambda > 0$, the bounds are explicit enough to derive a numerical α_0 up to which unique solvability holds. Unfortunately, the estimates we use are rather poor as they mostly neglect the right behavior of the density of X_t at the boundary, thus yielding a non-optimal value. Anyhow, they serve as a starting point for proving a refined estimate of the gradient of the density at the boundary: this is the required ingredient for proving that, at any time t, the mass of X_t decays linearly in the neighbourhood of 1, uniformly in t in compact sets, and thus to apply the existence and uniqueness argument in small time. Such a refined estimate is proven on any interval where a solution does exist: below, it is thus referred to as an *a priori* estimate. Coupled with the result of existence and uniqueness in small time, it permits to prove by induction that existence and uniqueness hold on any finite interval and thus on the whole of $[0, \infty)$.

The layout of the paper is as follows. In the first section below we make the necessary definitions and describe the setting in which we will be working. Section 3 is devoted to proving the existence and uniqueness of a solution to (1.2) in small time. Long-time *a priori* estimates are proven in Section 4. This section contains the crux of our argument, since we here show that, for α small enough, if there exists a solution, it does not exhibit the blow-up property highlighted in [3]. In Section 5 the final step is made, whereby we show that under the *a priori* Lipschitz bound, an unique solution exists to (1.2) that does not blow-up at any time. We conclude the paper in Section 6 with a discussion of the critical values of α in both the Brownian and the Ornstein-Uhlenbeck cases.

2. Setting

As stated in the introduction, we are interested in solutions to the nonlinear McKean-Vlasov-type SDE

$$X_t = X_0 + \int_0^t b(X_s)ds + \alpha\mathbb{E}(M_t) + W_t - M_t, \qquad t \geqslant 0, \tag{2.1}$$

where $X_0 < 1$ almost surely, $\alpha \in (0, 1)$ and W_t is a standard Brownian motion with respect to a filtration $(\mathcal{F}_t)_{t \geqslant 0}$ satisfying the usual conditions, with values in $\mathbb{R}$ and starting from 0. The jumps, or resets, of the system are described by

$$M_t = \sum_{k \geqslant 1} \mathbf{1}_{[0,t]}(\tau_k), \tag{2.2}$$

where the sequence of stopping times $(\tau_k)_{k \geqslant 0}$ are defined by $\tau_0 = 0$ and

$$\tau_k = \inf\{t > \tau_{k-1} : X_{t-} \geqslant 1\}, \quad k \geqslant 1. \tag{2.3}$$

We assume that $b : (-\infty, 1] \to \mathbb{R}$ is Lipschitz continuous such that

$$|b(x)| \leqslant \Lambda(|x| + 1), \qquad |b(x) - b(y)| \leqslant K|x - y|, \qquad \forall x, y \in (-\infty, 1].$$

Remark 2.1. *Although we take the intensity of the noise in* (2.1) *to be* 1 *rather than a general* $\sigma > 0$ *as stated in the introduction, this results in no loss of generality. Indeed, by setting* $u = t/\sigma^2$ *we can rewrite* (2.1) *as*

$$X_{\sigma^2 u} = X_0 + \int_0^{\sigma^2 u} b(X_s)ds + \alpha\mathbb{E}(M^X_{\sigma^2 u}) + W_{\sigma^2 u} - M^X_{\sigma^2 u}, \qquad u \geqslant 0,$$

for an arbitrary $\sigma > 0$*, where we have emphasized the dependence of the process* M *on* X *by writing* $M = M^X$*. Now, by Brownian scaling we have that* $W_{\sigma^2 u} = \sigma\widetilde{W}_u$ *where* $\widetilde{W}$ *is another Brownian motion starting from* 0*. Set* $\widetilde{X}_u = X_{\sigma^2 u}$*. Then, since the number of times the process* X *hits the threshold before time* $\sigma^2 u$ *is equal to the number of time the process* $\widetilde{X}$ *hits the threshold before time* u*, we have that* $M^X_{\sigma^2 u} = M^{\widetilde{X}}_u$*. Hence it follows that*

$$\widetilde{X}_u = X_0 + \int_0^{\sigma^2 u} b(X_s)ds + \alpha\mathbb{E}(M^{\widetilde{X}}_u) + \sigma\widetilde{W}_u - M^{\widetilde{X}}_u, \qquad u \geqslant 0.$$

By a change of variable we then have

$$\widetilde{X}_u = X_0 + \int_0^{u} \sigma^2 b(\widetilde{X}_s)ds + \alpha\mathbb{E}(M^{\widetilde{X}}_u) + \sigma\widetilde{W}_u - M^{\widetilde{X}}_u, \qquad u \geqslant 0.$$

Thus, if we modify the function b *accordingly,* $\widetilde{X}$ *satisfies exactly the equation of interest, but now with an arbitrary noise intensity.*

As discussed in the introduction, the key point is to look for a solution for which $t \mapsto \mathbb{E}(M_t)$ is continuously differentiable, which would correspond to a solution whose derivative does not exhibit a finite time blow-up. To this end we introduce the following objects:

- *The space* $\mathcal{L}$: For $T, A \geqslant 0$, let

$$\mathcal{L}(T, A) = \left\{ e \in \mathcal{C}^1[0, T] : e(0) = 0,\ e(s) \leqslant e(t)\ \forall s \leqslant t, \sup_{0 \leqslant t \leqslant T} |e'(t)| \leqslant A \right\}$$

where as usual $\mathcal{C}^1[0, T]$ denotes the space of continuously differentiable functions on $[0, T]$.

- *The map* Γ*:* For $e \in \mathcal{L}(T, A)$ define

$$\Gamma(e)(t) := \mathbb{E}(M^e_t) \tag{2.4}$$

where

$$M^e_t = \sum_{k \geqslant 1} \mathbf{1}_{[0,t]}(\tau^e_k) \tag{2.5}$$

and $\tau^e_k = \inf\{t > \tau^e_{k-1} : X^e_{t-} \geqslant 1\}$, $\tau_0 = 0$ and (X^e_t, M^e_t) is a solution to

$$X^e_t = X_0 + \int_0^t b(X^e_s)ds + \alpha e(t) + W_t - M^e_t, \qquad t \geqslant 0, \qquad X_0 < 1. \tag{2.6}$$

Note that the solution is well defined as b is Lipschitz continuous. The map Γ is thus defined as a map from $\mathcal{L}(T,A)$ into the set of non-decreasing functions on $[0,T]$. It thus depends on A as its domain of definition depends on A; for this reason, it should be denoted by Γ^A. Anyhow, since the family $(\Gamma^A)_{A\geqslant 0}$ is consistent in the sense that, for any $A'\leqslant A$, the restriction of Γ^A to $\mathcal{L}(T,A')$ coincides with $\Gamma^{A'}$, we can get rid of the superscript A in Γ^A and then use the simpler notation Γ.

The idea is to look for a fixed point of Γ in a certain space, which means that we are to find a space that is kept invariant by Γ. With this in mind, the following *a priori* stability result for Γ will be useful, the proof of which we leave until the end of the section.

Proposition 2.2. *Given $T>0$, $e\in\mathcal{L}(T,A)$ and some real $a\geqslant\mathbb{E}[(X_0)_+]$ (where $(x)_+$ indicates the positive part of $x\in\mathbb{R}$), it holds that*

$$\big(\forall t\in[0,T],\ e(t)\leqslant g(t)\big)\Rightarrow\big(\forall t\in[0,T],\ \Gamma(e)(t)\leqslant g(t)\big), \tag{2.7}$$

with

$$g(t):=\frac{a+(4+\Lambda T^{1/2})t^{1/2}}{1-\alpha}\exp\left(\frac{2\Lambda t}{1-\alpha}\right).$$

With this in mind, we thus also introduce the space $\mathcal{H}$, followed by our definition of a solution to (2.1):

- *The space $\mathcal{H}$*: For an initial value $X_0<1$, let

$$\mathcal{H}(T,A,X_0)=\left\{e\in\mathcal{L}(T,A):e(t)\leqslant C_T(\sqrt{t}+\mathbb{E}|X_0|)e^{\theta t}\right\},$$

where C_T and θ are chosen according to Proposition 2.2, that is

$$C_T:=\frac{4+\Lambda T^{1/2}}{1-\alpha},\quad \theta:=\frac{2\Lambda}{1-\alpha}.$$

Equip $\mathcal{H}(T,A,X_0)$ with the norm $\|e\|_{\mathcal{H}(T,A,X_0)}=\|e\|_{\infty,T}+\|e'\|_{\infty,T}$. Then $\mathcal{H}(T,A,X_0)$ is a complete metric space, since it is a closed subspace of $\mathcal{C}^1[0,T]$. We then define:

Definition 2.3 (Solution to (2.1))**.** *The coupled process $(X_t,M_t)_{0\leqslant t\leqslant T}$ will be said to be a solution to* (2.1) *up until time T if $(M_t)_{0\leqslant t\leqslant T}$ satisfies* (2.2) *subject to* (2.3)*, $([0,T]\ni t\mapsto\mathbb{E}(M_t))\in\mathcal{H}(A,T,X_0)$ for some $A\geqslant 0$, and $(X_t)_{0\leqslant t\leqslant T}$ is a strong solution of* (2.1) *up until time T.*

With the above definitions, it is clear that, for any $A\geqslant 0$, any fixed point of Γ that belongs to $\mathcal{H}(A,T,X_0)$ provides a solution to (2.1), and conversely that, for any solution $(X_t,M_t)_{0\leqslant t\leqslant T}$ of (2.1), $([0,T]\ni t\mapsto\mathbb{E}(M_t))$ must be a fixed point of Γ in $\mathcal{H}(A,T,X_0)$ for A as in Definition 2.3.

2.1. Proof of Proposition 2.2:

Proof of Proposition 2.2: Fix $T>0$. We first note that we may write

$$M_t^e=\sup_{s\leqslant t}\big\lfloor\big(Z_s^e\big)_+\big\rfloor,\quad Z_t^e=X_t^e+M_t^e,\quad t\in[0,T], \tag{2.8}$$

Indeed, one can see that for $t \in [\tau^e_k, \tau^e_{k+1})$, $k \geqslant 0$,

$$\begin{aligned}\sup_{s \leqslant t} \lfloor (Z^e_s)_+ \rfloor &= \max_{0 \leqslant j \leqslant k-1} \left(\sup_{s \in [\tau^e_j, \tau^e_{j+1})} \lfloor (X^e_s + j)_+ \rfloor, \sup_{s \in [\tau^e_k, t)} \lfloor (X^e_s + k)_+ \rfloor \right) \\ &= \max_{0 \leqslant j \leqslant k-1} (j+1, k) = M^e_t,\end{aligned}$$

using the fact that $X^e_t < 1$ for all $t \geqslant 0$.

Then, given $t \in [0,T]$ such that $Z^e_t \geqslant 0$, let $\rho^e := \sup\{s \in [0,t] : Z^e_s < 0\}$ ($\sup \emptyset = 0$). Pay attention that ρ^e is not a stopping time. Then, for $s \in [\rho^e, t]$,

$$|b(X^e_s)| \leqslant \Lambda(1 + |X^e_s|) \leqslant \Lambda\big(1 + |Z^e_s| + M^e_s\big) = \Lambda\big(1 + (Z^e_s)_+ + M^e_s\big). \tag{2.9}$$

By (2.8), we know that $M^e_s \leqslant \sup_{0 \leqslant r \leqslant s} (Z^e_r)_+$. Therefore,

$$|b(X^e_s)| \leqslant \Lambda \left(1 + 2 \sup_{0 \leqslant r \leqslant s} (Z^e_r)_+ \right).$$

By (2.6), we obtain:

$$Z^e_t \leqslant Z^e_{\rho^e} + \Lambda \int_{\rho^e}^t \left(1 + 2 \sup_{0 \leqslant r \leqslant s} (Z^e_r)_+ \right) ds + \alpha e(t) + W_t - W_{\rho^e}. \tag{2.10}$$

If $\rho^e > 0$, then $Z^e_{\rho^e} = 0$ as, obviously, $(Z^e_s)_{0 \leqslant s \leqslant T}$ is a continuous process. If $\rho^e = 0$, then $Z^e_{\rho^e} = X_0 \geqslant 0$. Therefore,

$$(Z^e_t)_+ \leqslant (X_0)_+ + \Lambda \int_0^t \left(1 + 2 \sup_{0 \leqslant r \leqslant s} (Z^e_r)_+ \right) ds + \alpha e(t) + 2 \sup_{0 \leqslant s \leqslant t} |W_s|. \tag{2.11}$$

Obviously, the above inequality still holds if $Z^e_t \leqslant 0$. Therefore, taking the supremum in the left-hand side, taking the expectation and applying Doob's maximal inequality for martingales,

$$\mathbb{E}\left[\sup_{0 \leqslant s \leqslant t} (Z^e_s)_+ \right] \leqslant \mathbb{E}[(X_0)_+] + \Lambda \int_0^t \left(1 + 2\mathbb{E}\left[\sup_{0 \leqslant r \leqslant s} (Z^e_r)_+ \right] \right) ds + \alpha e(t) + 4t^{1/2}, \tag{2.12}$$

which shows that $\mathbb{E}[\sup_{0 \leqslant t \leqslant T} (Z^e_t)_+]$ is finite. In particular, if

$$\forall t \in [0,T], \quad e(t) \leqslant g(t), \quad \text{with } g(t) := \frac{\mathbb{E}[(X_0)_+] + (4 + \Lambda T^{1/2}) t^{1/2}}{1-\alpha} \exp\left(\frac{2\Lambda t}{1-\alpha} \right),$$

and R^e stands for the deterministic hitting time:

$$R^e := \inf \left\{ t \in [0,T] : \mathbb{E}\left[\sup_{0 \leqslant s \leqslant t} (Z^e_s)_+ \right] > g(t) \right\} \quad (\inf \emptyset = +\infty),$$

then, for any $t \in [0, R^e \wedge T]$,

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{0 \leqslant s \leqslant t} (Z^e_s)_+\right] \\
&\leqslant \mathbb{E}[(X_0)_+] + \Lambda \int_0^t \big(1 + 2g(s)\big) ds + \alpha g(t) + 4t^{1/2} \\
&< \big(\mathbb{E}[(X_0)_+] + (4 + \Lambda T^{1/2}) t^{1/2}\big) \left[1 + \int_0^t \frac{2\Lambda}{1-\alpha} \exp\left(\frac{2\Lambda s}{1-\alpha}\right) ds\right] + \alpha g(t) \\
&= (1-\alpha) g(t) + \alpha g(t) = g(t).
\end{aligned}
$$

Now, by the continuity of the paths of Z^e and by the finiteness of $\mathbb{E}[\sup_{0 \leqslant t \leqslant T} (Z^e_t)_+]$, we deduce that $\mathbb{E}[\sup_{0 \leqslant s \leqslant t} (Z^e_s)_+]$ is continuous in t. Therefore, if $R^e \leqslant T$, then $\mathbb{E}[\sup_{0 \leqslant s \leqslant R^e} (Z^e_s)_+]$ must be equal to $g(R^e)$, but, by the above inequalities, this sounds as a contradiction. By (2.8), this proves the announced bound when a in the statement exactly matches $\mathbb{E}[(X_0)_+]$. The proof is completed in the same way when $a > \mathbb{E}[(X_0)_+]$. $\square$

Remark 2.4. *In the case when $X_0 = 0$, we have that Γ exhibits the same stability on functions e such that $e(t) \leqslant C_T \sqrt{t} e^{\theta t}$.*

3. Existence and uniqueness in small time

The main result of this section is the following:

Theorem 3.1. *Suppose there exist $\beta, \epsilon > 0$ such that $\mathbb{P}(X_0 \in dx) \leqslant \beta(1-x)dx$ for any $x \in (1-\epsilon, 1]$ and that the resulting density of the law of X_0 on the interval $(1-\epsilon, 1]$ is differentiable at point 1. Then there exist constants $A_1 \geqslant 1$ and $T_1 \in (0, 1]$, depending upon X_0 through β, ϵ and $\mathbb{E}|X_0|$ (which is assumed to be finite) only, such that*

$$\Gamma\big(\mathcal{H}(A_1, T_1, X_0)\big) \subset \mathcal{H}(A_1, T_1, X_0).$$

Moreover, for all $e_1, e_2 \in \mathcal{H}(A_1, T_1, X_0)$,

$$\|\Gamma(e_1) - \Gamma(e_2)\|_{\mathcal{H}(A_1,T_1,X_0)} \leqslant \frac{1}{2} \|e_1 - e_2\|_{\mathcal{H}(A_1,T_1,X_0)}.$$

Hence there exists an unique fixed point of the restriction of Γ to $\mathcal{H}(A_1, T_1, X_0)$, which provides a solution to (2.1) according to Definition 2.3 up until time T_1.

Remark 3.2. *The assumption $\mathbb{P}(X_0 \in dx) \leqslant \beta(1-x)dx$, for $x \in (1-\epsilon, 1]$, says that the distribution of X_0 has a density on the interval $(1-\epsilon, 1]$, and that the density is bounded by $\beta(1-x)$. This is a key assumption to prove Thereom 3.1 since it prevents the hitting time of 1 by X from having a Dirac mass at the initial time 0. In the proof, it permits to show that $\Gamma(e)$ is Lipschitz continuous when e itself is Lipschitz continuous. By contrast, we feel that the proof could be written without requiring the density of X_0 at point 1 to be differentiable. Here the differentiability of the density forces $\Gamma(e)$ to be continuously differentiable at time 0 when e is continuously differentiable, as specified in the definition of $\mathcal{L}(T, A)$. This proves to be really convenient in the whole analysis, but does not affect the final result in Section 5.*

3.1. **PDEs.** Given $e \in \mathcal{L}(T,A)$, we have

$$\Gamma(e)(t) = \mathbb{E}(M_t^e) = \sum_{k \geqslant 1} \int_0^t \mathbb{P}\big(\tau_{k+1}^e \in (s,t] \,|\tau_k^e = s\big)\mathbb{P}(\tau_k^e \in ds) + \mathbb{P}(\tau_1^e \leqslant t),$$

where $\mathbb{P}(\tau_k^e \in ds)$ is a convenient abuse of notation for denoting the law of τ_k^e and $\mathcal{B}(\mathbb{R}) \ni A \mapsto \mathbb{P}(\tau_{k+1}^e \in A|\tau_k^e = s)$ stands for the conditional law of τ_{k+1}^e given $\tau_k^e = s$. Here $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra on $\mathbb{R}$. Observing that the solution X^e to (2.6) is a Markov process, we may write

$$\Gamma(e)(t) = \mathbb{E}(M_t^e) = \sum_{k \geqslant 1} \int_0^t \mathbb{P}\big(\tau_1^{e^{\sharp s}} \leqslant t - s|X_0^{e^{\sharp s}} = 0\big)\mathbb{P}(\tau_k^e \in ds) + \mathbb{P}(\tau_1^e \leqslant t), \quad (3.1)$$

where $e^{\sharp s}$ stands for the mapping $([0, T-s] \ni t \mapsto e(t+s) - e(s)) \in \mathcal{L}(T-s, A)$. In order to understand the behaviour of Γ, it will become clear that one must study the marginal density of the killed process

$$\frac{d}{dy}\mathbb{P}(X_t^e \in dy, t < \tau_1^e).$$

Note here we have used the notation of the Radon-Nikodym derivative to denote the density of the killed process with respect to the Lebesgue measure, which makes sense thanks to the following Lemma:

Lemma 3.3. *Given an initial condition X_0 satisfying the same assumptions as in the statement of Theorem 3.1 together with $e \in \mathcal{L}(T,A)$, consider the solution $(\chi_t)_{0 \leqslant t \leqslant T}$ of the SDE*

$$d\chi_t = b(\chi_t)dt + \alpha e'(t)dt + dW_t, \quad t \in [0,T] \ ; \quad \chi_0 = X_0, \quad (3.2)$$

together with the stopping time

$$\tau_1 = \inf\{t \in [0,T] : \chi_t \geqslant 1\}.$$

Then, for any $t \in (0,T]$, the measure

$$\mathcal{B}((-\infty,1]) \ni A \mapsto \mathbb{P}\big(\chi_t \in A, t < \tau_1\big)$$

has a density with respect to the Lebesgue measure. Here, $\mathcal{B}((-\infty,1])$ denotes the Borel subsets of $(-\infty,1]$. Pay attention that the density is not normalized. Denoting the density by

$$p(t,y) = \frac{d}{dy}\mathbb{P}\big(\chi_t \in dy, t < \tau_1\big), \quad t \in [0,T], \ y \leqslant 1, \quad (3.3)$$

it is continuous and continuously differentiable in y on $(0,T] \times (-\infty,1]$ and admits Sobolev derivatives of order 1 in t and of order 2 in y in any L^ς, $\varsigma \geqslant 1$, on any compact subset of $(0,T] \times (-\infty,1)$. When $\beta = 0$ in the statement of Theorem 3.1, i.e. $X_0 \leqslant 1-\epsilon$ a.s., it is actually continuous and continuously differentiable in y on any compact subset of $([0,T] \times (-\infty,1]) \setminus (\{0\} \times (-\infty, 1-\epsilon])$. Moreover, p satisfies the Kolmogorov equation:

$$\partial_t p(t,y) + \partial_y\big[\big(b(y) + \alpha e'(t)\big)p(t,y)\big] - \frac{1}{2}\partial^2_{yy}p(t,y) = 0, \quad t \in (0,T], \ y < 1, \quad (3.4)$$

with the Dirichlet boundary condition $p(t,1) = 0$ and the measure-valued initial condition $p(0,y)dy = \mathbb{P}(X_0 \in dy)$, both $p(t,y)$ and $\partial_y p(t,y)$ decaying to 0 as $y \to -\infty$. Finally, τ_1 has a density on $[0,T]$, given by

$$\frac{d}{dt}\mathbb{P}(\tau_1 \leqslant t) = -\frac{1}{2}\partial_y p(t,1), \quad t \in [0,T], \tag{3.5}$$

the mapping $[0,T] \ni t \mapsto \partial_y p(t,1)$ being continuous and its supremum norm being bounded in terms of T, α, A, β and b only.

Lemma 3.3 is quite standard. Anyhow, it is difficult to find a complete proof of it under our assumptions. For this reason, we provide a complete proof in appendix (Section 7). We can apply the above result to (3.1) to obtain:

Proposition 3.4. *Given an initial condition X_0 satisfying the same assumption as in the statement of Theorem 3.1 together with $e \in \mathcal{L}(T,A)$, the mapping $[0,T] \ni t \mapsto \Gamma(e)(t)$ is continuously differentiable. Moreover,*

$$\frac{d}{dt}\big[\Gamma(e)\big](t) = -\int_0^t \frac{1}{2}\partial_y p_e^{(0,s)}(t-s,1)\frac{d}{ds}\big[\Gamma(e)\big](s)ds - \frac{1}{2}\partial_y p_e(t,1), \quad t \in [0,T], \tag{3.6}$$

where p_e represents the density of the process X^e killed at 1 and $p_e^{(0,s)}$ represents the density of the process $X^{e\sharp s}$ killed at 1 with $X_0^{e\sharp s} = 0$.

Proof. We first check that $\Gamma(e)$ is Lipschitz continuous on $[0,T]$. Considering a finite difference in (3.1) and using (3.5), we get, for $t, t+h \in [0,T]$,

$$\begin{aligned}
\Gamma(e)(t+h) - \Gamma(e)(t) = &\sum_{k \geqslant 1} \int_t^{t+h} \mathbb{P}\big(\tau_1^{e\sharp s} \leqslant t+h-s | X_0^{e\sharp s} = 0\big)\mathbb{P}(\tau_k^e \in ds) \\
&- \frac{1}{2}\sum_{k \geqslant 1}\int_0^t \int_{t-s}^{t+h-s} \partial_y p_e^{(0,s)}(r,1) dr \mathbb{P}(\tau_k^e \in ds) \\
&- \frac{1}{2}\int_t^{t+h} \partial_y p_e(s,1)ds.
\end{aligned} \tag{3.7}$$

By Lemma 3.3, we can handle the two last terms in the above to find a constant $C > 0$ such that

$$\begin{aligned}
\Gamma(e)(t+h) - \Gamma(e)(t) \leqslant &\sum_{k \geqslant 1} \int_t^{t+h} \mathbb{P}\big(\tau_1^{e\sharp s} \leqslant t+h-s | X_0^{e\sharp s} = 0\big)\mathbb{P}(\tau_k^e \in ds) \\
&+ Ch\big(1 + \Gamma(e)(T)\big),
\end{aligned}$$

the last term in the right-hand side being finite thanks to Proposition 2.2. Moreover, from the continuous differentiability of e, we deduce that

$$\begin{aligned}
&\lim_{h \searrow 0} \sup_{0 \leqslant s \leqslant T-h} \mathbb{P}\big(\tau_1^{e\sharp s} \leqslant h | X_0^{e\sharp s} = 0\big) \\
&= \lim_{h \searrow 0} \sup_{0 \leqslant s \leqslant T-h} \mathbb{P}\left(\sup_{0 \leqslant r \leqslant h} Z_r^{e\sharp s} \geqslant 1 | X_0^{e\sharp s} = 0\right) = 0,
\end{aligned} \tag{3.8}$$

where $Z^{e\sharp_s}$ is given by (2.8), the second line following from (2.11). Therefore, there exists a mapping $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ matching 0 at 0 and continuous at 0 such that

$$\Gamma(e)(t+h) - \Gamma(e)(t) \leqslant \eta(h)\big[\Gamma(e)(t+h) - \Gamma(e)(t)\big] + Ch\big(1 + \Gamma(e)(T)\big).$$

Choosing h small enough, Lipschitz continuity easily follows.

As a consequence, we can divide both sides of (3.7) by h and then let h tend to 0. By (3.8), we have for a given $t \in [0,T)$,

$$\begin{aligned}
&\lim_{h \searrow 0} h^{-1} \sum_{k \geqslant 1} \int_t^{t+h} \mathbb{P}\big(\tau_1^{e\sharp_s} \leqslant t+h-s | X_0^{e\sharp_s} = 0\big)\mathbb{P}(\tau_k^e \in ds) \\
&\leqslant \lim_{h \searrow 0} \left[\sup_{0 \leqslant s \leqslant T-h} \mathbb{P}\big(\tau_1^{e\sharp_s} \leqslant h | X_0^{e\sharp_s} = 0\big) \frac{\Gamma(e)(t+h) - \Gamma(e)(t)}{h} \right] = 0.
\end{aligned}$$

Handling the second term in (3.7) by Lemma 3.3 and using the Lebesgue Dominated Convergence Theorem, we deduce that

$$\frac{d}{dt}\Gamma(e)(t) = -\sum_{k \geqslant 1} \int_0^t \frac{1}{2} \partial_y p_e^{(0,s)}(t-s,1)\mathbb{P}(\tau_k^e \in ds) - \frac{1}{2}\partial_y p_e(t,1).$$

By Lemma 3.3, we know that $\partial_y p_e^{(0,s)}(\cdot,1)$ and $\partial_y p_e(\cdot,1)$ are continuous (in t). This proves that $(d/dt)\Gamma(e)$ is continuous as well.

Formula (3.6) follows from the relationship

$$\Gamma(e)(t) = \sum_{k \geqslant 1} \int_0^t \mathbb{P}(\tau_k^e \in ds), \quad t \in [0,T]. \tag{3.9}$$

□

3.2. **Density estimates in short time.** Suppose that $X_0 = x_0 < 1$. Following the parametrix method in [8, Chapter 1], one can show that the solution to (3.4) (then denoted by $p_e^{x_0}$) is given by

$$p_e^{x_0}(t,y) = q(t,x_0,y) - \int_0^t \int_{-\infty}^1 \left(\alpha e'(s) + b(1)\right) \partial_z p_e^{x_0}(s,z) q(t-s,z,y) dz ds, \tag{3.10}$$

for $t \in (0,T]$ and $y < 1$, where $q(t,x,y)$ is a solution to the PDE

$$\left\{ \begin{array}{rl} \partial_t q(t,x,y) & = (1/2)\partial^2_{yy} q(t,x,y) - \partial_y \left[(b(y) - b(1))q(t,x,y)\right] \\ q(t,x,1) & = 0 \\ q(0,x,y) & = \delta_0(x-y) \end{array} \right. \tag{3.11}$$

on $[0,T] \times (-\infty,1] \times (-\infty,1]$. We will need the following bound on the behaviour of q.

Proposition 3.5. *For any $T > 0$ there exists a constant $B_T > 0$ such that*

$$|\partial_y q(t,x,y)| \leqslant \frac{B_T}{t} \exp\left(-\frac{|\xi_t^x - y|^2}{B_T t}\right), \tag{3.12}$$

and

$$|\partial_y q(t,x,y)| \leqslant \frac{B_T}{t} \exp\left(-\frac{|x-\xi^y_{-t}|^2}{B_T t}\right), \tag{3.13}$$

for all $t \in [0,T]$, $x, y \leqslant 1$, *where*

$$\xi^x_t = x + \int_0^t \widetilde{b}(\xi^x_r) dr, \quad t \in \mathbb{R}, \tag{3.14}$$

and $\widetilde{b}(v) = b(v) - b(1)$ *if* $v < 1$ *while* $\widetilde{b}(v) = -\widetilde{b}(2-v)$ *if* $v \geqslant 1$.

Proof. Let $(\widetilde{X}_t)_{t \geqslant 0}$ denote the solution of the SDE:

$$d\widetilde{X}_t = \widetilde{b}(\widetilde{X}_t)dt + dW_t. \tag{3.15}$$

We can then see that $q(t,x,y)$ is the transition density of $\widetilde{X}_t$ killed at 1 (equation (3.11) is the Fokker-Planck equation for the killed process, as shown in Lemma 3.3).

Note that $\widetilde{b}(1+v) = -\widetilde{b}(1-v)$ for $v \in \mathbb{R}$, so that $\widetilde{b}$ is odd with respect to 1. Therefore, when initialised at $\widetilde{X}_0 = 1$ the process $(\widetilde{X}_t - 1)_{0 \leqslant t \leqslant T}$ satisfies

$$\widetilde{X}_t - 1 = \int_0^t \widetilde{b}(\widetilde{X}_s - 1 + 1)ds + W_t, \qquad t \in [0,T]$$

and the process $(1 - \widetilde{X}_t)_{0 \leqslant t \leqslant T}$ satisfies

$$\begin{aligned} 1 - \widetilde{X}_t &= -\int_0^t \widetilde{b}(\widetilde{X}_s)ds - W_t = \int_0^t \widetilde{b}(2 - \widetilde{X}_s)ds - W_t \\ &= \int_0^t \widetilde{b}(1 - \widetilde{X}_s + 1)ds - W_t, \end{aligned}$$

which is the same equation. Hence $(1 - \widetilde{X}_t)_{0 \leqslant t \leqslant T}$ and $(\widetilde{X}_t - 1)_{0 \leqslant t \leqslant T}$ have the same distribution when $\widetilde{X}_0 = 1$. Therefore the reflection principle (see [17, Section I.13]) applies and

$$q(t,x,y) = \widetilde{q}(t,x,y) - \widetilde{q}(t,x,2-y), \qquad t \in [0,T], \quad x, y \leqslant 1, \tag{3.16}$$

where $\widetilde{q}(t,x,y)$ is the transition density of the non-killed process.

We thus need to estimate the transition density $\widetilde{q}$ of the non-killed process. By Theorem 1.1 of [6] it follows that

$$\widetilde{q}(t,x,y) \leqslant \frac{B_T}{\sqrt{t}} \exp\left(-\frac{|y-\xi^x_t|^2}{B_T t}\right), \quad t > 0,\ x, y \in \mathbb{R}, \tag{3.17}$$

where

$$\xi^x_t = x + \int_0^t \widetilde{b}(\xi^x_s) ds, \qquad t \geqslant 0.$$

We claim that a similar bound holds for $|\partial_y \widetilde{q}(t,x,y)|$, i.e.

$$|\partial_y \widetilde{q}(t,x,y)| \leqslant \frac{B_T}{t} \exp\left(-\frac{|y-\xi^x_t|^2}{B_T t}\right), \quad t > 0,\ x, y \in \mathbb{R}. \tag{3.18}$$

Before we prove (3.18), we show that it implies (3.12) and (3.13). By (3.16), we deduce that

$$|\partial_y q(t,x,y)| \leqslant \frac{B_T}{t}\left[\exp\left(-\frac{|y-\xi_t^x|^2}{B_T t}\right)+\exp\left(-\frac{|2-y-\xi_t^x|^2}{B_T t}\right)\right], \quad t>0,\ x,y\leqslant 1.$$

Since $\widetilde{b}(1)=0$, we have $\xi_t^1=1$ for any $t\geqslant 0$. Therefore, by the comparison principle for ODEs, it must hold $\xi_t^x\leqslant \xi_t^1=1$ for any $t\geqslant 0$ and $x\leqslant 1$. As a consequence, for $t>0$ and $x,y\leqslant 1$,

$$\begin{aligned}|y-\xi_t^x| = |(y-1)-(\xi_t^x-1)| &\leqslant |y-1|+|\xi_t^x-1|\\ &= 1-y+1-\xi_t^x = 2-y-\xi_t^x = |2-y-\xi_t^x|.\end{aligned}$$

Therefore, the largest exponential term in the above bound for $|\partial_y q(t,x,y)|$ is the first one. Equation (3.12) follows directly (adjusting B_T as necessary). Now, by Gronwall's Lemma and thanks to the Lipschitz property of b, we have:

$$\exp(-Ks)|\xi_t^x-y| \leqslant |\xi_{t-s}^x-\xi_{-s}^y| \leqslant \exp(Ks)|\xi_t^x-y| \tag{3.19}$$

for all $s\in[0,t]$, so that, choosing $s=t$, we get (3.13).

We now turn to the proof of (3.18). Without any loss of generality, we can assume $\widetilde{b}$ to be a twice continuously differentiable function with a bounded second-order derivative. Indeed, if we can prove that, in such a case, (3.18) holds with respect to a constant B_T that does not refer to the second-order differentiability of $\widetilde{b}$, then (3.18) holds in the original setting as well by a standard mollification argument. Since $\widetilde{q}$ is the transition probability of the solution to (3.15), it satisfies the Fokker-Planck equation

$$\partial_t\widetilde{q}(t,x,y) = \frac{1}{2}\partial^2_{yy}\widetilde{q}(t,x,y) - \widetilde{b}(y)\partial_y\widetilde{q}(t,x,y) - \widetilde{b}'(y)\widetilde{q}(t,x,y),$$

for $t\in[0,T]$ and $x,y\in\mathbb{R}$. Define $(\widetilde{Y}_t^y)_{t\geqslant 0}$ to be the solution of the SDE

$$\begin{cases} d\widetilde{Y}_t^y &= -\widetilde{b}(\widetilde{Y}_t^y)dt + dW_t,\\ \widetilde{Y}_0^y &= y.\end{cases} \tag{3.20}$$

By Theorem 39 [15, p. 312], we know that, a.s., the mapping $[0,+\infty)\times\mathbb{R}\ni(t,y)\mapsto Y_t^y$ is continuously differentiable with respect to y and that

$$\partial_y\widetilde{Y}_s^y = \exp\left(-\int_0^s \widetilde{b}'(\widetilde{Y}_u^y)du\right). \tag{3.21}$$

Moreover, by Lemma 3.3, we know that $\widetilde{q}$ is regular enough in order to apply the Itô-Krylov formula (see [12, Section II.10]) to $[0,t/2]\ni s\mapsto \widetilde{q}\left(t-s,x,\widetilde{Y}_s^y\right)$, which yields

$$\widetilde{q}(t,x,y) = \mathbb{E}\left[\widetilde{q}\left(t/2,x,\widetilde{Y}_{t/2}^y\right)\right] - \int_0^{t/2}\mathbb{E}\left[\widetilde{b}'(\widetilde{Y}_s^y)\widetilde{q}\left(t-s,x,\widetilde{Y}_s^y\right)\right]ds.$$

By the Malliavin-Bismut-Elworthy formula (see, for example, Theorem 2.1 of [7]),

$$\partial_y \widetilde{q}(t,x,y) = \frac{2}{t}\mathbb{E}\left[\widetilde{q}(t/2,x,\widetilde{Y}^y_{t/2})\int_0^{t/2}\partial_y \widetilde{Y}^y_s dW_s\right] - \int_0^{t/2}\frac{1}{s}\mathbb{E}\left[\widetilde{b}'(\widetilde{Y}^y_s)\widetilde{q}\left(t-s,x,\widetilde{Y}^y_s\right)\int_0^s \partial_y \widetilde{Y}^y_r dW_r\right]ds. \tag{3.22}$$

By (3.21), $|\partial_y \widetilde{Y}^y_s| \leqslant \exp(Ks)$. Thus using (3.17) we can compute for any $0 < s \leqslant t/2$ (where the constant B_T changes from line to line below)

$$\begin{aligned}
I(s,t) &:= \left|\frac{1}{s}\mathbb{E}\left[\widetilde{q}(t-s,x,\widetilde{Y}^y_s)\int_0^s \partial_y \widetilde{Y}^y_r dW_r\right]\right| \\
&\leqslant \frac{1}{s}\left[\mathbb{E}\left(\widetilde{q}^2(t-s,x,\widetilde{Y}^y_s)\right)\right]^{1/2}\left[\mathbb{E}\left(\int_0^s \partial_y \widetilde{Y}^y_r dW_r\right)^2\right]^{1/2} \\
&\leqslant \frac{B_T}{\sqrt{s}}\left[\mathbb{E}\left(\widetilde{q}^2(t-s,x,\widetilde{Y}^y_s)\right)\right]^{1/2} \\
&\leqslant \frac{B_T}{\sqrt{s}}\left[\frac{1}{t-s}\mathbb{E}\left(\exp\left(-\frac{|\widetilde{Y}^y_s-\xi^x_{t-s}|^2}{B_T(t-s)}\right)\right)\right]^{1/2} \\
&= \frac{B_T}{\sqrt{s}}\left[\frac{1}{t-s}\int_{\mathbb{R}}\exp\left(-\frac{|z-\xi^x_{t-s}|^2}{B_T(t-s)}\right)\mathbb{P}(\widetilde{Y}^y_s \in dz)\right]^{1/2}.
\end{aligned}$$

By an estimate similar to (3.17) but for the density of $\widetilde{Y}$, we deduce

$$\begin{aligned}
I(s,t) &\leqslant \frac{B_T}{\sqrt{s}}\left[\frac{1}{(t-s)\sqrt{s}}\int_{\mathbb{R}}\exp\left(-\frac{|z-\xi^x_{t-s}|^2}{B_T(t-s)}\right)\exp\left(-\frac{|z-\xi^y_{-s}|^2}{B_T s}\right)dz\right]^{1/2} \\
&\leqslant \frac{B_T}{\sqrt{s}}\left[\frac{1}{\sqrt{t-s}\sqrt{t}}\exp\left(-\frac{|\xi^x_{t-s}-\xi^y_{-s}|^2}{B_T t}\right)\right]^{1/2},
\end{aligned}$$

where, for the first line, we have used the fact the flow associated with the ODE driven by $-\widetilde{b}$ is nothing but the backward flow associated with the ODE driven by $\widetilde{b}$, i.e. $(\xi^x_{-t})_{t\geqslant 0}$ satisfies

$$\xi^x_{-t} = x - \int_0^t \widetilde{b}(\xi^x_{-s})ds, \quad t \geqslant 0,$$

and, to pass from the first to the second line, we have recognised a Gaussian convolution. By (3.19), we deduce that

$$I(s,t) \leqslant \frac{B_T}{\sqrt{s}}\left[\frac{1}{\sqrt{t-s}\sqrt{t}}\exp\left(-\frac{|\xi^x_t-y|^2}{B_T t}\right)\right]^{1/2},$$

from which (3.22) yields

$$
\begin{aligned}
|\partial_y \widetilde{q}(t,x,y)| &\leqslant \frac{B_T}{t}\exp\left(-\frac{|\xi_t^x-y|^2}{B_T t}\right) \\
&\qquad + B_T \exp\left(-\frac{|\xi_t^x-y|^2}{B_T t}\right)\int_0^{t/2}\frac{1}{\sqrt{s}}\left[\frac{1}{\sqrt{t-s}\sqrt{t}}\right]^{1/2} ds \\
&\leqslant \frac{B_T}{t}\exp\left(-\frac{|\xi_t^x-y|^2}{B_T t}\right) + \frac{B_T}{\sqrt{t}}\exp\left(-\frac{|\xi_t^x-y|^2}{B_T t}\right)\int_0^{t/2}\frac{1}{\sqrt{s}} ds.
\end{aligned}
$$

This proves (3.18). □

Lemma 3.6. *Let $T>0$. Suppose $X_0 = x_0 < 1$ and $e \in \mathcal{L}(T,A)$. Then there exists a constant $\kappa_1(T)$ (independent of A and x_0) which increases with T such that*

$$
\sup_{0\leqslant s\leqslant t}\left[\sqrt{s}\,\|\partial_y p_e^{x_0}(s,.) - \partial_y q(s,x_0,.)\|_\infty\right] \leqslant (\alpha A + |b(1)|)\kappa_1(T),
$$

for all non-negative $t \leqslant \min\{[(\alpha A + |b(1)|)\kappa_1(T)]^{-2}, T\}$.

Proof. For notational sake, define

$$
F_e^{x_0}(t,y) := \partial_y p_e^{x_0}(t,y) - \partial_y q(t,x_0,y).
$$

By (3.10), we see that

$$
\begin{aligned}
|F_e^{x_0}(t,y)| &\leqslant (\alpha A + |b(1)|)\int_0^t\int_{-\infty}^1 |\partial_z q(s,x_0,z)\partial_y q(t-s,z,y)| dz ds \\
&\quad + (\alpha A + |b(1)|)\int_0^t\int_{-\infty}^1 |\partial_z p_e^{x_0}(s,z) - \partial_z q(s,x_0,z)|\,|\partial_y q(t-s,z,y)|\, dz ds.
\end{aligned} \tag{3.23}
$$

Thus defining $\widetilde{A} := \alpha A + |b(1)|$ and allowing B_T to increase as necessary from line to line below, by Proposition 3.5 we see that for $t \leqslant T$

$$
\begin{aligned}
&|F_e^{x_0}(t,y)| \\
&\leqslant \widetilde{A}B_T\int_0^t\int_{-\infty}^1 s^{-1}(t-s)^{-1}\exp\left(-\frac{|z-\xi_s^{x_0}|^2}{B_T s}\right)\exp\left(-\frac{|z-\xi_{-(t-s)}^y|^2}{B_T(t-s)}\right) dz ds \\
&\quad + \widetilde{A}B_T\int_0^t (t-s)^{-1}\,\|F_e^{x_0}(s,\cdot)\|_\infty\int_{-\infty}^1 \exp\left(-\frac{|z-\xi_{-(t-s)}^y|^2}{B_T(t-s)}\right) dz ds.
\end{aligned}
$$

We can recognise the first term in the above as a Gaussian convolution. Thus

$$
\begin{aligned}
|F_e^{x_0}(t,y)| &\leqslant \frac{\widetilde{A}B_T}{\sqrt{t}}\int_0^t\frac{1}{\sqrt{s}\sqrt{t-s}}\exp\left(-\frac{|\xi_{-(t-s)}^y - \xi_s^{x_0}|^2}{B_T t}\right) ds \\
&\qquad + \widetilde{A}B_T\int_0^t\frac{\sqrt{s}}{\sqrt{t-s}\sqrt{s}}\,\|F_e^{x_0}(s,\cdot)\|_\infty\, ds \\
&\leqslant \widetilde{A}B_T\left(\frac{1}{\sqrt{t}}\exp\left(-\frac{|\xi_{-t}^y - x_0|^2}{B_T t}\right) + \sup_{0\leqslant s\leqslant t}\left[\sqrt{s}\,\|F_e^{x_0}(s,\cdot)\|_\infty\right]\right),
\end{aligned} \tag{3.24}
$$

where again we have used (3.19). Therefore, by multiplying both sides of the inequality by $\sqrt{t}$, we deduce

$$\sup_{0 \leqslant s \leqslant t} \left[\sqrt{s}\, \|F_e^{x_0}(s,\cdot)\|_\infty\right] \leqslant \widetilde{A}B_T + \sqrt{t}\widetilde{A}B_T \sup_{0 \leqslant s \leqslant t} \left[\sqrt{s}\, \|F_e^{x_0}(s,\cdot)\|_\infty\right],$$

We can conclude that, for $t \in [0,T]$ such that $t \leqslant [2\widetilde{A}B_T]^{-2} = [2(\alpha A + |b(1)|)B_T]^{-2}$, we have that

$$\sup_{0 \leqslant s \leqslant t} \left[\sqrt{s}\, \|\partial_y p_e^{x_0}(s,.) - \partial_y q(s,x_0,.)\|_\infty\right] \leqslant 2\widetilde{A}B_T. \tag{3.25}$$

□

Proposition 3.7. *Let $T > 0$. Suppose $X_0 = x_0 < 1$, and that $e \in \mathcal{L}(T,A)$. Then there exists a constant $\kappa_2(T)$ (independent of A and x_0) which increases with T such that*

$$|\partial_y p_e^{x_0}(t,y) - \partial_y q(t,x_0,y)| \leqslant \kappa_2(T)(\alpha A + |b(1)|)\frac{1}{\sqrt{t}} \exp\left(-\frac{|\xi_t^{x_0} - y|^2}{\kappa_2(T)t}\right),$$

for all $t \leqslant \min\{[(\alpha A + |b(1)|)\kappa_2(T)]^{-2}, T\}$ and $y \leqslant 1$, where $(\xi_t^{x_0})_{t \geqslant 0}$ is as in Proposition 3.5.

Proof. Throughout the proof, we will use the fact the mapping

$$\varphi : \mathbb{R} \ni x \mapsto \frac{\exp(x) - 1}{x} \quad (\varphi(0) = 1), \tag{3.26}$$

is non-decreasing. We will also use the same notation as in the proof of Lemma 3.6:

$$F_e^{x_0}(t,y) := \partial_y p_e^{x_0}(t,y) - \partial_y q(t,x_0,y).$$

First Step. As noted in (3.23), we have for $t \in [0,T]$

$$\begin{aligned} |F_e^{x_0}(t,y)| \leqslant \widetilde{A} &\int_0^t \int_{-\infty}^1 |\partial_z q(s,x_0,z)\partial_y q(t-s,z,y)| dz ds \\ &+ \widetilde{A} \int_0^t \int_{-\infty}^1 |F_e^{x_0}(t_1,y_1)|\, |\partial_y q(t-t_1,y_1,y)|\, dy_1 dt_1 \end{aligned} \tag{3.27}$$

where we have denoted $\widetilde{A} = \alpha A + |b(1)|$. The fact that φ given by (3.26) is non-decreasing says that, for $t \in [0,T]$,

$$\frac{1 - \exp(-2KT)}{2KT} \leqslant \frac{1 - \exp(-2Kt)}{2Kt} \leqslant 1 \leqslant \frac{\exp(2Kt) - 1}{2Kt} \leqslant \frac{\exp(2KT) - 1}{2KT},$$

from which, together with Proposition 3.5, we deduce there exists $B_T > 0$ such that

$$|\partial_y q(t,x,y)| \leqslant \frac{B_T}{1 - e^{-2Kt}} \exp\left(-\frac{|x - \xi_{-t}^y|^2}{2B_T(1 - e^{-2Kt})}\right) \tag{3.28}$$

$$|\partial_y q(t,x,y)| \leqslant \frac{B_T}{e^{2Kt} - 1} \exp\left(-\frac{|\xi_t^x - y|^2}{2B_T(e^{2Kt} - 1)}\right). \tag{3.29}$$

Unlike in the previous lemma, B_T is now fixed. We then introduce the kernel

$$\mathcal{K}(t,x,y) = \begin{cases} \dfrac{1}{\sqrt{B_T(e^{2Kt}-1)}} \exp\left(-\dfrac{|\xi_t^x - y|^2}{2B_T(e^{2Kt}-1)}\right) & \text{if } t>0, \\ \dfrac{1}{\sqrt{B_T(1-e^{2Kt})}} \exp\left(-\dfrac{|x-\xi_t^y|^2}{2B_T(1-e^{2Kt})}\right) & \text{if } t<0. \end{cases} \tag{3.30}$$

For $0 < s < t \leqslant T$, $\mathcal{K}$ satisfies the Gaussian convolution property:

$$\begin{aligned} \mathcal{K}(s,x,\cdot) \otimes \mathcal{K}\big(-(t-s),\cdot,y\big) &:= \int_{\mathbb{R}} \mathcal{K}(s,x,z)\mathcal{K}\big(-(t-s),z,y\big)dz \\ &= \frac{\sqrt{2\pi}}{\sqrt{B_T(e^{2Ks}-e^{-2K(t-s)})}} \exp\left(-\frac{|\xi_s^x - \xi_{-(t-s)}^y|^2}{2B_T(e^{2Ks}-e^{-2K(t-s)})}\right). \end{aligned}$$

By (3.19), we know that $|\xi_t^x - y| \leqslant e^{K(t-s)}|\xi_s^x - \xi_{-(t-s)}^y|$ for all $0 < s < t$. Therefore,

$$\begin{aligned} \mathcal{K}(s,x,\cdot) \otimes \mathcal{K}\big(-(t-s),\cdot,y\big) &\leqslant \frac{\sqrt{2\pi}}{\sqrt{B_T(1-e^{-2Kt})}} \exp\left(-\frac{|\xi_t^x - y|^2}{2B_T(e^{2Kt}-1)}\right) \\ &\leqslant \sqrt{2\pi}e^{KT}\mathcal{K}(t,x,y). \end{aligned} \tag{3.31}$$

By (3.27), (3.28) and (3.29), we deduce that, for $t \in [0,T]$,

$$\begin{aligned} |F_e^{x_0}(t,y)| &\leqslant \widetilde{A}B_T^3 e^{KT}\mathcal{K}(t,x_0,y) \int_0^t \frac{\sqrt{2\pi}}{\sqrt{e^{2Ks}-1}\sqrt{1-e^{-2K(t-s)}}}ds \\ &\quad + \widetilde{A}B_T^{3/2} \int_0^t \int_{-\infty}^1 \frac{|F_e^{x_0}(t_1,y_1)|}{\sqrt{1-e^{-2K(t-t_1)}}} \mathcal{K}(-(t-t_1),y_1,y)dy_1dt_1. \end{aligned} \tag{3.32}$$

Using (3.26), we notice that

$$\begin{aligned} \int_0^t \frac{1}{\sqrt{e^{2Ks}-1}\sqrt{1-e^{-2K(t-s)}}}ds &\leqslant \frac{\sqrt{2KT}}{\sqrt{1-\exp(-2KT)}} \int_0^t \frac{1}{\sqrt{2Ks}\sqrt{2K(t-s)}}ds \\ &= \frac{\sqrt{T}}{\sqrt{2K(1-\exp(-2KT))}} \int_0^t \frac{1}{\sqrt{s}\sqrt{t-s}}ds \\ &\leqslant \frac{4\sqrt{T}}{\sqrt{2K(1-\exp(-2KT))}} =: C_T, \end{aligned} \tag{3.33}$$

using $\int_0^t (\sqrt{s}\sqrt{t-s})^{-1}ds = \int_0^1 (\sqrt{u}\sqrt{1-u})^{-1}du = \pi \leqslant 4$. Finally, (3.32) yields

$$\begin{aligned} |F_e^{x_0}(t,y)| &\leqslant \widetilde{A}\widetilde{C}_T B_T^3 \mathcal{K}(t,x_0,y) \\ &\quad + \widetilde{A}B_T^{3/2} \int_0^t \int_{-\infty}^1 \frac{|F_e^{x_0}(t_1,y_1)|}{\sqrt{1-e^{-2K(t-t_1)}}} \mathcal{K}(-(t-t_1),y_1,y)dy_1dt_1, \end{aligned} \tag{3.34}$$

with $\widetilde{C}_T := \sqrt{2\pi}C_T\exp(KT)$.

Second Step. We now prove by induction, that for any $N \geqslant 0$,

$$|F_e^{x_0}(t,y)| \leqslant \widetilde{A}\widetilde{C}_T B_T^3 \mathcal{K}(t,x_0,y) \sum_{i=0}^{N} \left(\sqrt{K}\widetilde{A}\widetilde{C}_T B_T^{3/2}\sqrt{t}\right)^i + R_{N+1}(t,y), \tag{3.35}$$

where

$$R_N(t,y) = (\widetilde{A}B_T^{3/2})^N \int_{0 \leqslant t_N \leqslant \ldots \leqslant t_0} dt_N \ldots dt_1 \prod_{i=0}^{N-1} \frac{1}{\sqrt{1-e^{-2K(t_i - t_{i+1})}}}$$
$$\int_{(-\infty,1]^N} dy_N \ldots dy_1 \, |F_e^{x_0}(t_N,y_N)| \prod_{i=0}^{N-1} \mathcal{K}\big(-(t_i - t_{i+1}), y_{i+1}, y_i\big),$$

with the convention $t_0 = t$ and $y_0 = y$. In the first step, we established (3.35) when $N = 0$. Assume now that, for some $N \geqslant 0$, (3.35) holds for any $t \in [0,T]$ and $y \leqslant 1$. Then, plugging the induction assumption at rank N into (3.34), we get:

$$\begin{aligned}
&|F_e^{x_0}(t,y)| \\
&\leqslant \widetilde{A}\widetilde{C}_T B_T^3 \mathcal{K}(t,x_0,y) \\
&\quad + \widetilde{A}\widetilde{C}_T B_T^3 \left(\sum_{i=0}^{N} \left(\sqrt{K}\widetilde{A}\widetilde{C}_T B_T^{3/2}\sqrt{t}\right)^i \right) \widetilde{A}B_T^{3/2} \int_0^t \frac{\mathcal{K}(t_1,x_0,\cdot) \otimes \mathcal{K}(-(t-t_1),\cdot,y)}{\sqrt{1-e^{-2K(t-t_1)}}} dt_1 \\
&\quad + R_{N+2}(t,y).
\end{aligned}$$

By (3.31), we deduce that

$$\begin{aligned}
&|F_e^{x_0}(t,y)| \\
&\leqslant \widetilde{A}\widetilde{C}_T B_T^3 \mathcal{K}(t,x_0,y) \\
&\quad + \widetilde{A}\widetilde{C}_T B_T^3 \left(\sum_{i=0}^{N} \left(\sqrt{K}\widetilde{A}\widetilde{C}_T B_T^{3/2}\sqrt{t}\right)^i \right) \widetilde{A}B_T^{3/2} \sqrt{2\pi} e^{KT} \mathcal{K}(t,x_0,y) \int_0^t \frac{1}{\sqrt{1-e^{-2K(t-t_1)}}} dt_1 \\
&\quad + R_{N+2}(t,y).
\end{aligned}$$

Following (3.33), we have:

$$\begin{aligned}
\int_0^t \frac{ds}{\sqrt{1-e^{-2K(t-s)}}} &\leqslant \frac{\sqrt{2KT}}{\sqrt{1-\exp(-2KT)}} \int_0^t \frac{1}{\sqrt{2K(t-s)}} ds \\
&= \frac{\sqrt{T}}{\sqrt{1-\exp(-2KT)}} \int_0^t \frac{1}{\sqrt{t-s}} ds \\
&\leqslant \frac{2\sqrt{t}\sqrt{T}}{\sqrt{1-\exp(-2KT)}} \leqslant \sqrt{t}\sqrt{K}C_T.
\end{aligned}$$

Since $\widetilde{C}_T = \sqrt{2\pi}\exp(KT)C_T$, we deduce that (3.35) holds at rank $N+1$.

Third Step. Suppose that $\sqrt{t} \leqslant [\sqrt{K}\widetilde{A}\widetilde{C}_T B_T^{3/2}]^{-1}/2$. Then, the series in (3.35) is convergent (and bounded above by 2). The point is thus to prove that $R_N(t,y) \to 0$

as $N \to \infty$ for all $y \leqslant 1$. By Lemma 3.6, we deduce that, for $\sqrt{t} \leqslant [\widetilde{A}\kappa_1(T)]^{-1}$,

$$\begin{aligned} &R_N(t,y) \\ &\leqslant (\sqrt{2\pi}\widetilde{A}B_T^{3/2})^N \widetilde{A}\kappa_1(T) \int_{0 \leqslant t_N \leqslant t_{N-1} \leqslant \ldots \leqslant t_0} \frac{1}{\sqrt{t_N}} \prod_{i=0}^{N-1} \frac{1}{\sqrt{1-e^{-2K(t_i-t_{i+1})}}} dt_1 \ldots dt_N. \end{aligned}$$

Using (3.26), we obtain

$$\begin{aligned} R_N(t,y) \leqslant &\left(\sqrt{2\pi}\widetilde{A}B_T^{3/2} \frac{\sqrt{T}}{\sqrt{1-e^{-2KT}}} \right)^N \widetilde{A}\kappa_1(T) \\ &\times \int_{0 \leqslant t_N \leqslant t_{N-1} \leqslant \ldots \leqslant t_0} \frac{1}{\sqrt{t_N}} \prod_{i=0}^{N-1} \frac{1}{\sqrt{t_i - t_{i+1}}} dt_1 \ldots dt_N. \end{aligned} \tag{3.36}$$

To compute the integral in the right-hand side of the above, define

$$J_n(t) = \int_0^t u^{(n-1)/2}(t-u)^{-1/2} du, \quad t \geqslant 0, \ n \in \mathbb{N}.$$

By the change of variable $u = ts$, we have $J_n(t) = t^{n/2} J_n(1)$. Note that $J_0(1) = \pi \leqslant 4$. Moreover, for $n \geqslant 1$,

$$J_n(1) \leqslant n^{1/2} \int_0^{1-1/n} u^{(n-1)/2} du + \int_{1-1/n}^1 (1-u)^{-1/2} du \leqslant \frac{2n^{1/2}}{n+1} + 2n^{-1/2} \leqslant 4n^{-1/2}. \tag{3.37}$$

With this notation

$$\begin{aligned} I_{N,0}(t) &:= \int_{0 \leqslant t_N \leqslant t_{N-1} \leqslant \ldots \leqslant t_0} \frac{1}{\sqrt{t_N}} \prod_{i=0}^{N-1} \frac{1}{\sqrt{t_i - t_{i+1}}} dt_1 \ldots dt_N \\ &= \int_{0 \leqslant t_{N-1} \leqslant \ldots \leqslant t_0} J_0(t_{N-1}) \prod_{i=0}^{N-2} \frac{1}{\sqrt{t_i - t_{i+1}}} dt_1 \ldots dt_{N-1}, \end{aligned}$$

where the second equality holds for $N \geqslant 2$. More generally, setting for any $n \leqslant N-2$

$$I_{N,n}(t) := \int_{0 \leqslant t_{N-(n+1)} \leqslant \ldots \leqslant t_0} J_n(t_{N-(n+1)}) \prod_{i=0}^{N-(n+2)} \frac{1}{\sqrt{t_i - t_{i+1}}} dt_1 \ldots dt_{N-(n+1)},$$

we have, for any $0 \leqslant n \leqslant N-3$,

$$\begin{aligned}
I_{N,n}(t) &= J_n(1) \int_{0 \leqslant t_{N-(n+1)} \leqslant \dots \leqslant t_0} t_{N-(n+1)}^{n/2} \prod_{i=0}^{N-(n+2)} \frac{1}{\sqrt{t_i - t_{i+1}}} dt_1 \dots dt_{N-(n+1)} \\
&= J_n(1) \int_{0 \leqslant t_{N-(n+2)} \leqslant \dots \leqslant t_0} \left[\int_0^{t_{N-(n+2)}} \frac{t_{N-(n+1)}^{n/2}}{\sqrt{t_{N-(n+2)} - t_{N-(n+1)}}} dt_{N-(n+1)} \right] \\
&\qquad \times \prod_{i=0}^{N-(n+3)} \frac{1}{\sqrt{t_i - t_{i+1}}} dt_1 \dots dt_{N-(n+2)} \\
&= J_n(1) I_{N,n+1}(t),
\end{aligned}$$

so that, by (3.37),

$$\begin{aligned}
I_{N,0}(t) &= \left[\prod_{n=0}^{N-3} J_n(1) \right] I_{N,N-2}(t) = \left[\prod_{n=0}^{N-2} J_n(1) \right] J_{N-1}(t) \\
&= \left[\prod_{n=0}^{N-1} J_n(1) \right] t^{(N-1)/2} \leqslant 4[4t^{1/2}]^{N-1} [(N-1)!]^{-1/2},
\end{aligned}$$

using the fact that $I_{N,N-2}(t) = J_{N-2}(1) J_{N-1}(t)$. Going back to (3.36), we deduce that $R_N(t,y) \to 0$ as $N \to \infty$. Hence taking the limit as $N \to \infty$ in (3.35) yields (with $\sqrt{t} \leqslant [\sqrt{K} \widetilde{A} \widetilde{C}_T B_T^{3/2}]^{-1}/2$)

$$|F_e^{x_0}(t,y)| \leqslant 2 \widetilde{A} \widetilde{C}_T B_T^3 \mathcal{K}(t, x_0, y),$$

from which the result follows. □

Complementary to the above two results, we have the following two, which estimate the difference between the densities of two processes driven by two different drifts.

Lemma 3.8. *Let $T > 0$. Suppose $X_0 = x_0 < 1$, and that $e_1, e_2 \in \mathcal{L}(T, A)$. Then there exists a constant $\kappa_3(T)$ (independent of A and x_0) which increases with T such that*

$$\sup_{0 \leqslant s \leqslant t} \left[\sqrt{s} \left\| \partial_y p_{e_1}^{x_0}(s, .) - \partial_y p_{e_2}^{x_0}(s, .) \right\|_\infty \right] \leqslant \widetilde{A} \kappa_3(T) \|e_1' - e_2'\|_{\infty,t},$$

for all $t \leqslant \min\{[\widetilde{A} \kappa_3(T)]^{-2}, T\}$, where $\widetilde{A} := \max(\alpha A + |b(1)|, 1)$.

Proof. Let $y \leqslant 1$. Again by (3.10), we see that

$$\begin{aligned}
&\left| \partial_y p_{e_1}^{x_0}(t,y) - \partial_y p_{e_2}^{x_0}(t,y) \right| \\
&\leqslant (\alpha A + |b(1)|) \int_0^t \int_{-\infty}^1 \left| \partial_z p_{e_1}^{x_0}(s,z) - \partial_z p_{e_2}^{x_0}(s,z) \right| |\partial_y q(t-s,z,y)| \, dz ds \\
&\quad + \alpha \|e_1' - e_2'\|_{\infty,t} \int_0^t \int_{-\infty}^1 \left| \partial_z p_{e_2}^{x_0}(s,z) - \partial_z q(s,x_0,z) \right| |\partial_y q(t-s,z,y)| \, dz ds \\
&\quad + \alpha \|e_1' - e_2'\|_{\infty,t} \int_0^t \int_{-\infty}^1 |\partial_z q(s,x_0,z)| \, |\partial_y q(t-s,z,y)| \, dz ds. \qquad (3.38)
\end{aligned}$$

Using Proposition 3.5 once again and the notation (3.30) (see also (3.28) and (3.29)) together with (3.26), we see that

$$\begin{aligned}
&\left|\partial_y p_{e_1}^{x_0}(t,y) - \partial_y p_{e_2}^{x_0}(t,y)\right| \\
&\leqslant \widetilde{A}\kappa_3(T) \sup_{0\leqslant s\leqslant t}\left[\sqrt{s}\left\|\partial_z p_{e_1}^{x_0}(s,.) - \partial_z p_{e_2}^{x_0}(s,.)\right\|_\infty\right] \\
&\quad + \kappa_3(T)\left\|e_1' - e_2'\right\|_{\infty,t}\int_0^t\int_{-\infty}^1 \frac{\left|\partial_z p_{e_2}^{x_0}(s,z) - \partial_z q(s,x_0,z)\right|}{\sqrt{t-s}}\mathcal{K}(-(t-s),z,y)dzds \\
&\quad + \kappa_3(T)\left\|e_1' - e_2'\right\|_{\infty,t}\int_0^t\int_{-\infty}^1 \frac{1}{\sqrt{s}}\frac{1}{\sqrt{t-s}}\mathcal{K}(s,x_0,z)\mathcal{K}(-(t-s),z,y)dzds,
\end{aligned}$$

for some constant $\kappa_3(T) > 0$, where $\widetilde{A} = \max(\alpha A + |b(1)|, 1)$. Using Proposition 3.7 and (3.31), and allowing the constant $\kappa_3(T)$ to increase as necessary from line to line below, it follows that

$$\begin{aligned}
&\left|\partial_y p_{e_1}^{x_0}(t,y) - \partial_y p_{e_2}^{x_0}(t,y)\right| \\
&\leqslant \widetilde{A}\kappa_3(T) \sup_{0\leqslant s\leqslant t}\left[\sqrt{s}\left\|\partial_y p_{e_1}^{x_0}(s,.) - \partial_y p_{e_2}^{x_0}(s,.)\right\|_\infty\right] \\
&\quad + \kappa_3(T)\widetilde{A}\left\|e_1' - e_2'\right\|_{\infty,t}\int_0^t\int_{-\infty}^1 \frac{1}{\sqrt{t-s}}\mathcal{K}(s,x_0,z)\mathcal{K}(-(t-s),z,y)dzds \\
&\quad + \kappa_3(T)\left\|e_1' - e_2'\right\|_{\infty,t}\mathcal{K}(t,x_0,y),
\end{aligned}$$

for all $t \leqslant \min\{[\widetilde{A}\kappa_2(T)]^{-2}, T\}$. Using (3.31) again, together with the bound $\widetilde{A} \geqslant 1$, we deduce:

$$\begin{aligned}
\left|\partial_y p_{e_1}^{x_0}(t,y) - \partial_y p_{e_2}^{x_0}(t,y)\right| &\leqslant \widetilde{A}\kappa_3(T) \sup_{0\leqslant s\leqslant t}\left[\sqrt{s}\left\|\partial_y p_{e_1}^{x_0}(s,.) - \partial_y p_{e_2}^{x_0}(s,.)\right\|_\infty\right] \\
&\quad + \kappa_3(T)\widetilde{A}\left\|e_1' - e_2'\right\|_{\infty,t}\mathcal{K}(t,x_0,y). \qquad (3.39)
\end{aligned}$$

Assuming that $\kappa_3(T) \geqslant \kappa_2(T)$ and multiplying the above inequality by $\sqrt{t}$, we deduce that, for $t \leqslant \min\{[2\widetilde{A}\kappa_3(T)]^{-2}, T\}$ we have

$$\begin{aligned}
&\sup_{0\leqslant s\leqslant t}\left[\sqrt{s}\left\|\partial_y p_{e_1}^{x_0}(s,.) - \partial_y p_{e_2}^{x_0}(s,.)\right\|_\infty\right] \\
&\qquad \leqslant 2\kappa_3(T)\sup_{0<t\leqslant T}\left[\frac{\sqrt{t}}{\sqrt{B_T(\exp(2Kt)-1)}}\right]\widetilde{A}\left\|e_1' - e_2'\right\|_{\infty,t}.
\end{aligned}$$

□

Proposition 3.9. *Let $T > 0$. Suppose $X_0 = x_0 < 1$, and that $e_1, e_2 \in \mathcal{L}(T, A)$. Then there exists a constant $\kappa_4(T)$ (independent of A and x_0) which increases with T such that*

$$\left|\partial_y p_{e_1}^{x_0}(t,y) - \partial_y p_{e_2}^{x_0}(t,y)\right| \leqslant \widetilde{A}\kappa_4(T)\frac{1}{\sqrt{t}}\exp\left(-\frac{|\xi_t^{x_0} - y|^2}{\kappa_4(T)t}\right)\left\|e_1' - e_2'\right\|_{\infty,t},$$

for all $t \leqslant \min\{[\widetilde{A}\kappa_4(T)]^{-2}, T\}$ and $y \leqslant 1$, where $(\xi_t^{x_0})_{t\geqslant 0}$ is as in Proposition 3.5 and $\widetilde{A} := \max(\alpha A + |b(1)|, 1)$.

Proof. We follow the strategy of the proof of Proposition 3.7. We thus define

$$G^{x_0}_{e_1,e_2}(t,y) = \partial_y p^{x_0}_{e_1}(t,y) - \partial_y p^{x_0}_{e_2}(t,y).$$

Going back to (3.38) we can proceed in the same way as in Lemma 3.8 in order to bound the second two terms of this expression. We then get that, for some constant $\kappa_4(T) > 0$,

$$\begin{aligned}\left|G^{x_0}_{e_1,e_2}(t,y)\right| &\leqslant \widetilde{A}\kappa_4(T)\,\|e_1' - e_2'\|_{\infty,t}\,\mathcal{K}(t,x_0,y)\\ &\quad + \widetilde{A}\kappa_4(T)\int_0^t\int_{-\infty}^1 \frac{\left|G^{x_0}_{e_1,e_2}(t_1,y_1)\right|}{\sqrt{1-e^{-2K(t-t_1)}}}\mathcal{K}(-(t-t_1),y_1,y)dy_1dt_1.\end{aligned}$$

We can iterate this inequality in exactly the same way as in Proposition 3.7 (precisely, we can divide both sides by $\|e_1' - e_2'\|_{\infty,t}$ in order to recover (3.34) with $|F^{x_0}_e(\cdot,y)|$ replaced by $|G^{x_0}_{e_1,e_2}(\cdot,y)|/\|e_1'-e_2'\|_{\infty,t}$ therein and then follow (3.35)), and the convergence follows from Lemma 3.8. This yields the result. $\square$

3.3. **Conclusion.**

Proposition 3.10. *Let $T > 0$ and X_0 be as in the statement of Theorem 3.1 with (w.l.o.g.) $\beta \geqslant 1$. Suppose $e_1, e_2 \in \mathcal{H}(T,A,X_0)$. Then there exists a constant $\kappa(T)$, independent of A, β and ϵ, and increasing in T, and a constant $\widetilde{\kappa}(T,\beta,\epsilon)$, independent of A and increasing in T, such that*

$$\sup_{0\leqslant s\leqslant t}\left|\frac{d}{ds}\mathbb{E}(M^{e_1}_s) - \frac{d}{ds}\mathbb{E}(M^{e_2}_s)\right| \leqslant \widetilde{A}\widetilde{\kappa}(T,\beta,\epsilon)\sqrt{t}\,(1+\mathbb{E}|X_0|)\,\|e_1'-e_2'\|_{\infty,t},$$

for $t \leqslant \min\{[\widetilde{A}\kappa(T)]^{-2}, T\}$, where $\widetilde{A} := \max(\alpha A + |b(1)|, 1)$.

Proof. We have by (2.4) and (3.6)

$$\begin{aligned}&\left|\frac{d}{dt}\mathbb{E}(M^{e_1}_t) - \frac{d}{dt}\mathbb{E}(M^{e_2}_t)\right|\\ &\quad\leqslant \frac{1}{2}\int_{-\infty}^1 \left|\partial_y p^x_{e_1}(t,1) - \partial_y p^x_{e_2}(t,1)\right|\mathbb{P}(X_0\in dx)\\ &\qquad + \frac{1}{2}\sum_{k\geqslant 1}\int_0^t \left|\partial_y p^{(0,s)}_{e_1}(t-s,1) - \partial_y p^{(0,s)}_{e_2}(t-s,1)\right|\mathbb{P}(\tau^{e_1}_k\in ds) \qquad (3.40)\\ &\qquad + \frac{1}{2}\int_0^t \left|\partial_y p^{(0,s)}_{e_2}(t-s,1)\right|\left|\frac{d}{ds}\mathbb{E}(M^{e_1}_s) - \frac{d}{ds}\mathbb{E}(M^{e_2}_s)\right|ds\\ &\quad := \frac{1}{2}\big(L_1+L_2+L_3\big).\end{aligned}$$

Suppose $t \leqslant T$ and $\sqrt{t} \leqslant [\widetilde{A}\kappa_4(T)]^{-1}$, where $\kappa_4(T)$ is as in Proposition 3.9 and $\widetilde{A} = \max(\alpha A + |b(1)|, 1)$. Considering the first term only, we can use this result to

see that

$$\begin{aligned}L_1 &\leqslant \widetilde{A}\beta\kappa_4(T)\left(\int_{1-\epsilon}^1 \frac{1}{\sqrt{t}}\exp\left(-\frac{(1-x)^2}{\kappa_4(T)t}\right)(1-x)dx\right)\|e_1'-e_2'\|_{\infty,t}\\ &\quad+\widetilde{A}\kappa_4(T)\left(\int_{-\infty}^{1-\epsilon}\frac{1}{\sqrt{t}}\exp\left(-\frac{(1-x)^2}{\kappa_4(T)t}\right)\mathbb{P}(X_0\in dx)\right)\|e_1'-e_2'\|_{\infty,t},\end{aligned}$$

where we have used the bi-Lipschitz property (3.19) in the statement of Proposition 3.9 (and thus modified the value of $\kappa_4(T)$ accordingly), together with the fact that $\xi_t^1 = 1$ for all $t \geqslant 0$. Indeed, by definition (see the statement of Proposition 3.5), we have

$$\xi_t^1 = 1 + \int_0^t \widetilde{b}(\xi_s^1)ds, \quad \widetilde{b}(1) = 0.$$

We deduce that there exists a constant $\kappa_5(T,\beta,\epsilon) > 0$, which is independent of A and which is allowed to increase as necessary from line to line below, such that (the value of $\kappa_4(T)$ being also allowed to increase from line to line)

$$\begin{aligned}L_1 &\leqslant \widetilde{A}\beta\kappa_4(T)\sqrt{t}\left(\int_0^\infty z\exp\left(-\frac{z^2}{\kappa_4(T)}\right)dz\right)\|e_1'-e_2'\|_{\infty,t}\\ &\quad+\widetilde{A}\kappa_4(T)\frac{1}{\sqrt{t}}\exp\left(-\frac{\epsilon^2}{\kappa_4(T)t}\right)\|e_1'-e_2'\|_{\infty,t}\\ &\leqslant \widetilde{A}\kappa_5(T,\beta,\epsilon)\sqrt{t}\|e_1'-e_2'\|_{\infty,t}.\end{aligned}\tag{3.41}$$

We can then use Proposition 3.9 again to see that

$$L_2 \leqslant \widetilde{A}\kappa_4(T)\sup_{0<s\leqslant t}\left[s^{-1/2}\exp\left(-\frac{1}{\kappa_4(T)s}\right)\right]\mathbb{E}\big(M_t^{e_1}\big)\|e_1'-e_2'\|_{\infty,t}.$$

By Proposition 2.2, we deduce that

$$L_2 \leqslant \widetilde{A}\kappa_4(T)\sqrt{t}\big(\sqrt{t}+\mathbb{E}|X_0|\big)\|e_1'-e_2'\|_{\infty,t},\tag{3.42}$$

where we have used the elementary inequality $\exp(-1/v) \leqslant v$ for all $v \geqslant 0$. We finally turn to L_3 in (3.40). By Proposition 3.7, Proposition 3.5 and (3.19), we also have that

$$\begin{aligned}\big|\partial_y p_{e_2}^{(0,s)}(t-s,1)\big| &\leqslant \kappa_4(T)\widetilde{A}\frac{1}{\sqrt{t-s}}\exp\left(-\frac{1}{\kappa_4(T)(t-s)}\right)+|\partial_y q(t-s,0,1)|\\ &\leqslant \kappa_4(T)\exp\left(-\frac{1}{\kappa_4(T)(t-s)}\right)\left[\frac{\widetilde{A}}{\sqrt{t-s}}+\frac{1}{t-s}\right]\\ &\leqslant \kappa_4(T)\widetilde{A}.\end{aligned}\tag{3.43}$$

Thus, from (3.40), (3.41), (3.42) and (3.43), we deduce

$$\left|\frac{d}{dt}\mathbb{E}(M_t^{e_1}) - \frac{d}{dt}\mathbb{E}(M_t^{e_2})\right| \leqslant \widetilde{A}\kappa_5(T,\beta,\epsilon)\sqrt{t}(1+\mathbb{E}|X_0|)\|e_1' - e_2'\|_{\infty,t} + \widetilde{A}\kappa_4(T)\int_0^t \left|\frac{d}{ds}\mathbb{E}(M_s^{e_1}) - \frac{d}{ds}\mathbb{E}(M_s^{e_2})\right| ds.$$

By taking the supremum over all $s \leqslant t$ in the above, we have, for $t \leqslant (2\kappa_4(T)\widetilde{A})^{-1}$ (which actually follows from the aforementioned condition $t \leqslant (\kappa_4(T)\widetilde{A})^{-2}$ by assuming w.l.o.g. $\kappa_4(T) \geqslant 2$),

$$\sup_{0\leqslant s\leqslant t}\left|\frac{d}{ds}\mathbb{E}(M_s^{e_1}) - \frac{d}{ds}\mathbb{E}(M_s^{e_2})\right| \leqslant 2\widetilde{A}\kappa_5(T,\beta,\epsilon)\sqrt{t}\,(1+\mathbb{E}|X_0|)\,\|e_1' - e_2'\|_{\infty,t}.$$

□

Proof. **[Proof of Theorem 3.1]** Choose

$$A_1 = 2\sup_{0\leqslant t\leqslant 1}\left|\frac{d}{dt}\mathbb{E}(M_t^0)\right| + 1,$$

and set

$$\widetilde{A}_1 = \max(\alpha A_1 + |b(1)|, 1).$$

Note that A_1 and $\widetilde{A}_1$ depend on β. Then choose $T_1 \leqslant \min\{[\widetilde{A}_1\kappa(1)]^{-2}, 1\}$ such that

$$\sqrt{T_1}\widetilde{\kappa}(1,\beta,\epsilon)\widetilde{A}_1(1+\mathbb{E}|X_0|) \leqslant \frac{1}{4}, \tag{3.44}$$

where $\kappa(1)$ and $\widetilde{\kappa}(1,\beta,\epsilon)$ are as in the Proposition 3.10. By that result, if $e \in \mathcal{H}(A_1,T_1,X_0)$ then

$$\left|\frac{d}{dt}\mathbb{E}(M_t^e)\right| \leqslant \sqrt{t}\widetilde{\kappa}(T_1,\beta,\epsilon)\widetilde{A}_1(1+\mathbb{E}|X_0|)A_1 + \left|\frac{d}{dt}\mathbb{E}(M_t^0)\right|$$

for all $t \leqslant \min\{[\widetilde{A}_1\kappa(T_1)]^{-2}, 1\}$. By definition, we have $T_1 \leqslant 1$ so that $\kappa(T_1) \leqslant \kappa(1)$ and $\widetilde{\kappa}(T_1,\beta,\epsilon) \leqslant \widetilde{\kappa}(1,\beta,\epsilon)$. Therefore

$$\left|\frac{d}{dt}\mathbb{E}(M_t^e)\right| \leqslant \sqrt{t}\widetilde{\kappa}(1,\beta,\epsilon)\widetilde{A}_1(1+\mathbb{E}|X_0|)A_1 + \left|\frac{d}{dt}\mathbb{E}(M_t^0)\right|$$

for all $t \leqslant \min\{[\widetilde{A}_1\kappa(1)]^{-2}, 1\}$. Hence for all $t \in [0,T_1]$

$$\left|\frac{d}{dt}\mathbb{E}(M_t^e)\right| \leqslant \sqrt{t}\widetilde{\kappa}(1,\beta,\epsilon)\widetilde{A}_1(1+\mathbb{E}|X_0|)A_1 + \left|\frac{d}{dt}\mathbb{E}(M_t^0)\right| \leqslant \frac{A_1}{2} + \sup_{0\leqslant t\leqslant 1}\left|\frac{d}{dt}\mathbb{E}(M_t^0)\right| \leqslant A_1$$

by (3.44), so that $\Gamma(e) \in \mathcal{H}(A_1,T_1,X_0)$.

To prove that Γ is a contraction on $\mathcal{H}(A_1,T_1,X_0)$, first note that for $e \in \mathcal{H}(A_1,T_1,X_0)$

$$\|e'\|_{\infty,T_1} \leqslant \|e\|_{\mathcal{H}(A,T_1,X_0)} \leqslant 2\|e'\|_{\infty,T_1}$$

by the mean value theorem, since $e(0) = 0$ and $T_1 \leqslant 1$. Thus for any $e_1, e_2 \in \mathcal{H}(A_1, T_1, X_0)$

$$\begin{aligned}
\|\Gamma(e_1) - \Gamma(e_2)\|_{\mathcal{H}(A_1,T_1,X_0)} &\leqslant 2 \|\Gamma(e_1)' - \Gamma(e_2)'\|_{\infty,T_1} \\
&\leqslant 2 \sup_{t \leqslant T_1} \left| \frac{d}{dt}\mathbb{E}(M_t^{e_1}) - \frac{d}{dt}\mathbb{E}(M_t^{e_2}) \right| \\
&\leqslant 2\sqrt{T_1}\widetilde{\kappa}(T_1, \beta, \epsilon)\left(1 + \mathbb{E}|X_0|\right)\widetilde{A}_1 \|e_1' - e_2'\|_{\infty,T_1} \\
&\leqslant \frac{1}{2}\|e_1 - e_2\|_{\mathcal{H}(A_1,T_1,X_0)},
\end{aligned}$$

by our choice of T_1 and using Proposition 3.10 once more. Since $\mathcal{H}(A_1, T_1, X_0)$ is a closed subspace of $\mathcal{C}^1[0,T]$ (a complete metric space), the existence of a fixed point for Γ follows from the Banach Fixed Point Theorem. □

4. Long-time estimates

In order to extend the existence and uniqueness from small time to any arbitrarily prescribed interval, we need an a priori bound for the Lipschitz constant of $t \mapsto \mathbb{E}(M_t)$ on any finite interval $[0,T]$. In the whole section, for a given initial condition $X_0 = x_0 \in (-\infty, 1)$, we thus assume that there exists a solution to (2.1) according to Definition 2.3 i.e. such that $e : [0,T] \ni t \mapsto \mathbb{E}(M_t)$ is continuously differentiable. The aim is to then prove a universal bound for the Lipschitz constant of $[0,T] \ni t \mapsto \mathbb{E}(M_t)$:

Theorem 4.1. *For a given $\epsilon \in (0,1)$, there exists a positive constant $\alpha_0 \in (0,1]$, only depending upon ϵ, the Lipschitz constant K and the linear growth constant Λ of b, such that, for any $\alpha \in (0, \alpha_0)$ and any positive time $T > 0$, there exist a constant $\mathcal{M}_T$, only depending on T, ϵ, K and Λ, such that, for any initial condition $X_0 = x_0 \leqslant 1 - \epsilon$, any solution to (2.1) according to Definition 2.3 satisfies*

$$\forall t \in [0,T], \quad e'(t) = \frac{d}{dt}\mathbb{E}(M_t) \leqslant \mathcal{M}_T.$$

Remark 4.2. *In the proof below, we provide an expression for α_0 (or put it differently a lower bound for the best possible choice of α_0): we refer to the statement of Proposition 4.5, where the constant α_0 therein is the same as the one in the statement of Theorem 4.1. Without any further assumptions on the coefficients, the expression in Proposition 4.5 is not completely explicit as it depends upon a non-explicit constant. However, it becomes tractable when $b(x) = -\lambda x$, which is the common case treated in the neuroscience literature. We thus refer the reader to Subsection 6.1 for a detailed discussion about the critical threshold α_0 in this framework.*

4.1. Reformulation of the equation and *a priori* bounds for the solution.

In the whole proof, we shall use a reformulated version of the nonlinear equation (2.1). Indeed, given a solution $(X_t, M_t)_{0 \leqslant t \leqslant T}$ to (2.1) on some interval $[0,T]$ according to Definition 2.3, we set

$$Z_t = X_t + M_t, \quad t \in [0,T]. \tag{4.1}$$

Then, the dynamics of $(Z_t)_{0 \leqslant t \leqslant T}$ have the form:

$$Z_t = X_0 + \int_0^t b(X_s)ds + \alpha \mathbb{E}(M_t) + W_t, \quad t \in [0, T], \tag{4.2}$$

so that, by continuity of the mapping $[0, T] \ni t \mapsto e(t) = \mathbb{E}(M_t)$, the process $(Z_t)_{0 \leqslant t \leqslant T}$ has continuous paths. Moreover, as noted above in Proposition 2.2, M_t is completely defined in terms of Z_t, since it holds that

$$M_t = \lfloor \big(\sup_{0 \leqslant s \leqslant t} Z_s \big)_+ \rfloor = \sup_{0 \leqslant s \leqslant t} \lfloor (Z_s)_+ \rfloor. \tag{4.3}$$

Following the proof of Proposition 2.2, we claim:

Lemma 4.3. *There exists a constant $B(T, \alpha, b)$, only depending upon T, α, b and non-decreasing in α, such that*

$$\sup_{0 \leqslant t \leqslant T} e(t) = e(T) \leqslant \mathbb{E} \left[\sup_{0 \leqslant t \leqslant T} (Z_t)_+ \right] \leqslant B(T, \alpha, b). \tag{4.4}$$

A possible choice for B is

$$B(T, \alpha, b) = \frac{\mathbb{E}[(X_0)_+] + 4T^{1/2} + \Lambda T}{1 - \alpha} \exp \left(\frac{2\Lambda T}{1 - \alpha} \right).$$

Proof. The proof is a straightforward consequence of (2.12) in the proof of Proposition 2.2. □

4.2. **Local Hölder bound of the solution.** We now turn to the critical point of the proof. Indeed, in the next subsection, we shall prove that, for α small enough, the function $t \mapsto e(t) = \mathbb{E}(M_t)$ generated by some solution to (2.1) according to Definition 2.3 (so that e is continuously differentiable) satisfies an a priori 1/2-Hölder bound, with an explicit Hölder constant. This acts as the keystone of the argument to extend the local existence and uniqueness result into a global one. As a first step, the proof consists in establishing a local Hölder bound for e in the case when the probability that the process X lies in the neighbourhood of 1 is not too large.

Lemma 4.4. *Keep the same notation as in the previous subsection and consider in particular a solution $(X_t)_{0 \leqslant t \leqslant T}$ to (2.1) on some interval $[0, T]$, with $T > 0$ and initial condition $X_0 = x_0$. Assume in addition that there exist some time $t_0 \in [0, T]$ and two constants $\epsilon \in (0, 1)$ and $c \in (0, 1/\alpha)$ such that $x_0 \leqslant 1 - \epsilon$, and, for any Borel subset $A \subset [1 - \epsilon, 1]$,*

$$\mathbb{P}(X_{t_0} \in A) \leqslant c|A|, \tag{4.5}$$

where $|A|$ stands for the Lebesgue measure of A. Then, with

$$\mathcal{B}_0 = \frac{\exp(2\Lambda)[(8 + 5c + 8\epsilon^{-1})\Lambda + 4(2 + c + \epsilon^{-1})]}{1 - c\alpha},$$

it holds that, for any $h \in (0, 1)$,

$$\left. \begin{array}{l} \mathcal{B}_0 \exp(2\Lambda h) h^{1/2} \leqslant \epsilon/2 \\ t_0 + h \leqslant T \end{array} \right\} \Rightarrow e(t_0 + h) - e(t_0) \leqslant \mathcal{B}_0 h^{1/2}.$$

Proof. Without any loss of generality, we can assume $t_0 = 0$, with T being understood as $T - t_0$. Indeed, setting

$$X_t^{\sharp t_0} := X_{t_0+t}, \quad t \in [0, T - t_0], \tag{4.6}$$

we observe that, for $t \in [0, T - t_0]$,

$$X_t^{\sharp t_0} = X_{t_0} + \int_0^t b(X_r^{\sharp t_0}) dr + \alpha \mathbb{E}(M_{t+t_0} - M_{t_0}) + W_{t+t_0} - W_{t_0} - \bigl(M_{t+t_0} - M_{t_0}\bigr). \tag{4.7}$$

Here $M_{t+t_0} - M_{t_0}$ represents the number of times the process X reaches 1 within the interval $(t_0, t+t_0]$. Therefore, this also matches the number of times the process $X^{\sharp t_0}$ hits 1 within the interval $(0, t]$, so that $X^{\sharp t_0}$ indeed satisfies the nonlinear equation (2.1) on $[0, T - t_0]$, with $X_0^{\sharp t_0} = X_{t_0}$ as initial condition and with respect to the shifted Brownian motion $(W_t^{\sharp t_0} := W_{t_0+t} - W_{t_0})_{0 \leqslant t \leqslant T - t_0}$. In what follows, t_0 is thus assumed to be zero, the new T standing for the previous $T - t_0$ and the new X_0 matching the previous X_{t_0} and thus satisfying (4.5).

For a given $h \in (0, 1)$, such that $h \leqslant T$, and a given $\mathcal{B}_0 > 0$ (the value of which will be fixed later), we then define the deterministic hitting time:

$$R = \inf\bigl\{t \in [0, h] : \mathbb{E}(M_t) = e(t) \geqslant \mathcal{B}_0 h^{1/2}\bigr\}.$$

Following the proof of (2.11) (see more specifically (2.9)), we have, for any $t \in [0, h \wedge R]$,

$$\begin{aligned} M_t \leqslant \sup_{0 \leqslant s \leqslant t} (Z_s)_+ &\leqslant (X_0)_+ + \Lambda \int_0^t \bigl(1 + (Z_s)_+ + M_s\bigr) ds + \alpha e(t) + 2 \sup_{0 \leqslant s \leqslant t} |W_s| \\ &\leqslant (X_0)_+ + 2\Lambda \int_0^t \bigl(1 + M_s\bigr) ds + \alpha \mathcal{B}_0 h^{1/2} + 2 \sup_{0 \leqslant s \leqslant t} |W_s| \\ &\leqslant (X_0)_+ + 2\Lambda h + 2\Lambda \int_0^t M_s ds + \alpha \mathcal{B}_0 h^{1/2} + 2 \sup_{0 \leqslant s \leqslant t} |W_s|, \end{aligned}$$

where we have used (4.3) to pass from the first to the second line. By Gronwall's Lemma, we obtain

$$\begin{aligned} M_t &\leqslant \exp(2\Lambda h) \left[(X_0)_+ + 2\Lambda h + \alpha \mathcal{B}_0 h^{1/2} + 2 \sup_{0 \leqslant s \leqslant h} |W_s|\right] \\ &\leqslant (X_0)_+ + \exp(2\Lambda h) \left[4\Lambda h + \alpha \mathcal{B}_0 h^{1/2} + 2 \sup_{0 \leqslant s \leqslant h} |W_s|\right], \end{aligned} \tag{4.8}$$

as $\exp(2\Lambda h) \leqslant 1 + 2\Lambda h \exp(2\Lambda h)$ and $(X_0)_+ \leqslant 1$.

Assume that $\mathcal{B}_0 \exp(2\Lambda h) h^{1/2} \leqslant \epsilon/2 \leqslant 1/2$. Then, by Doob's L^2 inequality for martingales,

$$\begin{aligned} \sum_{k \geqslant 2} \mathbb{P}(M_t \geqslant k) &\leqslant \sum_{k \geqslant 2} \mathbb{P}\left(\exp(2\Lambda h) \left[4\Lambda h + 2 \sup_{0 \leqslant s \leqslant h} |W_s|\right] \geqslant k - 3/2\right) \\ &\leqslant 2 \exp(2\Lambda h) \mathbb{E}\bigl[4\Lambda h + 4|W_h|\bigr] \leqslant \exp(2\Lambda h)\bigl[8\Lambda h + 8h^{1/2}\bigr]. \end{aligned} \tag{4.9}$$

Moreover,

$$\begin{aligned}
&\mathbb{P}(M_t \geqslant 1) \\
&\leqslant \mathbb{P}\left((X_0)_+ + \exp(2\Lambda h) \left[4\Lambda h + \alpha \mathcal{B}_0 h^{1/2} + 2 \sup_{0 \leqslant s \leqslant h} |W_s| \right] \geqslant 1 \right) \\
&\leqslant \mathbb{P}\left(X_0 \in [1-\epsilon, 1], X_0 + \exp(2\Lambda h) \left[4\Lambda h + \alpha \mathcal{B}_0 h^{1/2} + 2 \sup_{0 \leqslant s \leqslant h} |W_s| \right] \geqslant 1 \right) \\
&\quad + \mathbb{P}\left(\exp(2\Lambda h) \left[4\Lambda h + 2 \sup_{0 \leqslant s \leqslant h} |W_s| \right] \geqslant \epsilon/2 \right) \\
&:= I_1 + I_2,
\end{aligned}$$

where we have used $\mathcal{B}_0 \exp(2\Lambda h) h^{1/2} \leqslant \epsilon/2$ in the third line.

By Doob's maximal inequality, we deduce that

$$I_2 \leqslant 2\exp(2\Lambda h)\epsilon^{-1}\mathbb{E}\big[4\Lambda h + 2|W_h|\big] \leqslant \exp(2\Lambda h)\epsilon^{-1}\big[8\Lambda h + 4h^{1/2}\big]. \tag{4.10}$$

We now switch to I_1. By independence of X_0 and $(W_s)_{0 \leqslant s \leqslant T}$ and by (4.5),

$$\begin{aligned}
I_1 &\leqslant c \int_0^{\epsilon} \mathbb{P}\left(\exp(2\Lambda h) \left[4\Lambda h + \alpha \mathcal{B}_0 h^{1/2} + 2 \sup_{0 \leqslant s \leqslant h} |W_s| \right] \geqslant x \right) dx \\
&\leqslant c \int_0^{+\infty} \mathbb{P}\left(\exp(2\Lambda h) \left[4\Lambda h + \alpha \mathcal{B}_0 h^{1/2} + 2 \sup_{0 \leqslant s \leqslant h} |W_s| \right] \geqslant x \right) dx \\
&= c \exp(2\Lambda h) \mathbb{E}\left[4\Lambda h + \alpha \mathcal{B}_0 h^{1/2} + 2 \sup_{0 \leqslant s \leqslant h} |W_s| \right].
\end{aligned}$$

By Doob's L^2 inequality,

$$I_1 \leqslant c \exp(2\Lambda h)\big[4\Lambda h + \alpha \mathcal{B}_0 h^{1/2} + 4h^{1/2}\big].$$

Together with (4.10), we deduce that

$$\mathbb{P}(M_t \geqslant 1) \leqslant \exp(2\Lambda h)\big[4(c + 2\epsilon^{-1})\Lambda h + 4(c + \epsilon^{-1})h^{1/2} + c\alpha \mathcal{B}_0 h^{1/2}\big].$$

From (4.9), we finally obtain, for $t \leqslant R \wedge h$,

$$\begin{aligned}
\mathbb{E}(M_t) &= \sum_{k \geqslant 1} \mathbb{P}(M_t \geqslant k) \\
&\leqslant \exp(2\Lambda h)\big[4(2 + c + 2\epsilon^{-1})\Lambda h + 4(2 + c + \epsilon^{-1})h^{1/2} + c\alpha \mathcal{B}_0 h^{1/2}\big] \\
&\leqslant \exp(2\Lambda h)\big[(8 + 5c + 8\epsilon^{-1})\Lambda h + 4(2 + c + \epsilon^{-1})h^{1/2}\big] + c\alpha \mathcal{B}_0 h^{1/2},
\end{aligned}$$

provided $\mathcal{B}_0 \exp(2\Lambda h) h^{1/2} \leqslant \epsilon/2 \leqslant 1/2$, which implies

$$c\alpha \mathcal{B}_0 \exp(2\Lambda h) h^{1/2} \leqslant c\alpha \mathcal{B}_0 h^{1/2} + c\Lambda h,$$

using the fact that $\exp(2\Lambda h) \leqslant 1 + 2\Lambda h \exp(2\Lambda h)$. Therefore, if $R \leqslant h$, then we can choose $t = R$ in the left-hand side above. By continuity of e on $[0, T]$, it then holds $e(R) = \mathcal{B}_0 h^{1/2}$, so that

$$\begin{aligned}
(1 - c\alpha)\mathcal{B}_0 h^{1/2} &\leqslant \exp(2\Lambda h)\big[(8 + 5c + 8\epsilon^{-1})\Lambda h + 4(2 + c + \epsilon^{-1})h^{1/2}\big] \\
&< \exp(2\Lambda)\big[(8 + 5c + 8\epsilon^{-1})\Lambda + 4(2 + c + \epsilon^{-1})\big]h^{1/2},
\end{aligned}$$

which is not possible when

$$\mathcal{B}_0 = \frac{\exp(2\Lambda)[(8+5c+8\epsilon^{-1})\Lambda + 4(2+c+\epsilon^{-1})]}{1-c\alpha}.$$

Precisely, with $\mathcal{B}_0$ as above and $\mathcal{B}_0 \exp(2\Lambda h) h^{1/2} \leqslant \epsilon/2$ it cannot hold $R \leqslant h$. $\square$

4.3. **Global Hölder bound.** In this subsection, we shall prove:

Proposition 4.5. *For a given $\epsilon \in (0,1)$, there exists a positive constant $\alpha_0 \in (0,1]$, only depending upon ϵ, K and Λ, such that, for any $\alpha < \alpha_0$, there exists a constant $\mathcal{B}$, only depending on α, ϵ, K and Λ, such that, for any positive time $T > 0$ and any initial condition $X_0 = x_0 \leqslant 1-\epsilon$, any solution to (2.1) according to Definition 2.3, so that the mapping $e : [0,T] \ni t \mapsto \mathbb{E}(M_t)$ is continuously differentiable, satisfies, for any $h \in (0,1)$ and $t_0 \in [0,T]$,*

$$\left.\begin{array}{l} \mathcal{B}h^{1/2} \leqslant \epsilon/2 \\ t_0 + h \leqslant T \end{array}\right\} \Rightarrow e(t_0+h) - e(t_0) \leqslant \mathcal{B}h^{1/2}.$$

Note that $\mathcal{B}$ above may differ from $\mathcal{B}_0$ in the statement of Lemma 4.4. The constant α_0 can be described as follows. Defining T_0 as the largest time less than 1 such that

$$\big(1-\epsilon\big)\exp(\Lambda T_0) \leqslant 1 - 7\epsilon/8, \quad \Lambda T_0 \exp(\Lambda T_0) \leqslant \epsilon/8,$$

then α_0 can be chosen as the largest (positive) real satisfying (with $B(T_0,\alpha_0,b)$ as in Lemma 4.3)

$$\begin{aligned}
&\alpha_0 B(T_0,\alpha_0,b) \leqslant \epsilon/4, \\
&\alpha_0 2^{3/2}(c')^{3/2} \exp\left(-\frac{1}{2}\right)\left[\epsilon^{-1} + B(T_0,\alpha_0,b)\right] \leqslant 1, \\
&\alpha_0 \left[c' T_0^{-1/2} + 2^{3/2}(c')^{3/2}\exp\left(-\frac{1}{2}\right) B(T_0,\alpha_0,b)\right] \leqslant 1,
\end{aligned}$$

where the constant c' is defined by the following property: $c' > 0$ depending on K only is such that for any diffusion process $(U_t)_{0 \leqslant t \leqslant 1}$ satisfying

$$dU_t = F(t,U_t)dt + dW_t, \quad t \in [0,1],$$

where $U_0 = 0$ and $F : [0,T] \times \mathbb{R} \to \mathbb{R}$ is K-Lipschitz in x and satisfies $F(t,0) = 0$ for any $t \in [0,1]$, it holds that

$$\frac{d}{dx}\mathbb{P}(U_t \in dx) \leqslant \frac{c'}{\sqrt{t}} \exp\left(-\frac{x^2}{c't}\right), \quad x \in \mathbb{R},\ t \in (0,1].$$

The proof relies on the following

Lemma 4.6. *Given an initial condition $X_0 = x_0 \leqslant 1-\epsilon$, with $\epsilon \in (0,1)$, and a solution $(X_t)_{0 \leqslant t \leqslant T}$ to (2.1) on some interval $[0,T]$, the random variable X_t has a density on $(-\infty,1]$, for any $t \in (0,T]$. Moreover, defining T_0 as in the statement of Proposition 4.5 and choosing $\alpha \leqslant \alpha_1$ satisfying*

$$\alpha_1 B(T_0,\alpha_1,b) \leqslant \epsilon/4,$$

then, for $x \in [1-\epsilon/4, 1)$,

$$\frac{d}{dx}\mathbb{P}(X_t \in dx) \leqslant 2^{3/2}(c')^{3/2}\exp\left(-\frac{1}{2}\right)\left[\epsilon^{-1} + B(T_0,\alpha,b)\right] \quad \text{if } t \leqslant T_0$$
$$\frac{d}{dx}\mathbb{P}(X_t \in dx) \leqslant c' T_0^{-1/2} + 2^{3/2}(c')^{3/2}\exp\left(-\frac{1}{2}\right) B(T_0,\alpha,b) \quad \text{if } t > T_0,$$

where the constant c' *is as in the statement of Proposition 4.5.*

Before we prove Lemma 4.6, we introduce some materials. As usual, we set $e(t) = \mathbb{E}(M_t)$, for $t \in [0,T]$, the mapping e being assumed to be continuously differentiable on $[0,T]$. Moreover, with $(X_t)_{0 \leqslant t \leqslant T}$, we associate the sequence of hitting times $(\tau_k)_{k \geqslant 0}$ given by (2.3) . We then investigate the marginal distributions of $(X_t)_{0 \leqslant t \leqslant T}$. Given a Borel subset $A \subset (-\infty, 1]$, we write in the same way as in the proof of (3.1):

$$\begin{aligned}\mathbb{P}(X_t \in A) &= \mathbb{P}(X_t \in A, \tau_1 > t) \\ &\quad + \sum_{k \geqslant 1} \int_0^t \mathbb{P}(X_t \in A, \tau_{k+1} > t | \tau_k = s)\mathbb{P}(\tau_k \in ds),\end{aligned} \tag{4.11}$$

where the notation $\mathbb{P}(\cdot|\tau_k = s)$ stands for the conditional law given $\tau_k = s$. Following (4.6) and (4.7), we can shift the system by length $s \in [0,T]$. Precisely, we know that $(X_r^{\sharp_s} := X_{s+r})_{0 \leqslant r \leqslant T-s}$ satisfies

$$X_r^{\sharp_s} = X_s + \int_0^r b\big(X_u^{\sharp_s}\big)du + \alpha e^{\sharp_s}(r) + W_{s+r} - W_s - M_r^{\sharp_s}, \tag{4.12}$$

with

$$\begin{aligned}&e^{\sharp_s}(r) := e(s+r) - e(s) \quad ; \quad M_r^{\sharp_s} := \sum_{k \geqslant 1} \mathbf{1}_{(0,r]}\big(\tau_k^{\sharp_s}\big) \quad ; \\ &\tau_k^{\sharp_s} := \inf\big\{u > \tau_{k-1}^{\sharp_s} : X_{s+u-} \geqslant 1\big\}, \quad k \geqslant 1 \quad ; \quad \tau_0^{\sharp_s} := 0.\end{aligned} \tag{4.13}$$

Conditionally on $\tau_k = s$, the law of $(X_r^{\sharp_s})_{0 \leqslant r \leqslant T-s}$ until $\tau_1^{\sharp_s}$ coincides with the law of $(\hat{Z}_r^{\sharp_s,0})_{0 \leqslant r \leqslant T-s}$ until the first time it reaches 1, where, for a given $\mathcal{F}_0$-measurable initial condition ζ with values in $(-\infty, 1)$, $(\hat{Z}_r^{\sharp_s,\zeta})_{0 \leqslant r \leqslant T-s}$ stands for the solution of the SDE:

$$\hat{Z}_r^{\sharp_s,\zeta} = \zeta + \int_0^r b\big(\hat{Z}_u^{\sharp_s,\zeta}\big)du + \alpha e^{\sharp_s}(r) + W_r, \quad r \in [0, T-s]. \tag{4.14}$$

Below, we will write $\hat{Z}_r^\zeta$ for $\hat{Z}_r^{\sharp_0,\zeta}$. In the end, we thus have (by (4.11)):

$$\begin{aligned}
\mathbb{P}(X_t \in A) &= \mathbb{P}(X_t \in A, \tau_1 > t) \\
&\qquad + \sum_{k \geqslant 1} \int_0^t \mathbb{P}(X_{t-s}^{\sharp_s} \in A, \tau_1^{\sharp_s} > t-s | \tau_k = s)\mathbb{P}(\tau_k \in ds) \\
&\leqslant \mathbb{P}(\hat{Z}_t^{X_0} \in A) + \sum_{k \geqslant 1} \int_0^t \mathbb{P}(\hat{Z}_{t-s}^{\sharp_s,0} \in A)\mathbb{P}(\tau_k \in ds) \\
&= \mathbb{P}(\hat{Z}_t^{X_0} \in A) + \int_0^t \mathbb{P}(\hat{Z}_{t-s}^{\sharp_s,0} \in A) e'(s) ds,
\end{aligned} \tag{4.15}$$

for any Borel set $A \subset (-\infty, 1]$, the passage from the third to the fourth line following from (3.9).

We can now turn to:

Proof of Lemma 4.6. Given an initial condition $x_0 \in (-\infty, 1-\epsilon]$ for $\varepsilon \in (0,1)$, we know from Delarue and Menozzi [6] that $\hat{Z}_t^{x_0}$ has a density for any $t \in (0,T]$ (and thus $\hat{Z}_{t-s}^{\sharp_s,0}$ as well for $0 \leqslant s < t$). From (4.15), we deduce that the law of X_t has a density on $(-\infty, 1]$ since $\mathbb{P}(X_t \in A) = 0$ when $|A| = 0$, where $|A|$ stands for the Lebesgue measure of A. Moreover, there exists a constant $c' \geqslant 1$, depending on K only, such that, for any $t \in [0, T \wedge 1]$:

$$\frac{d}{dx}\mathbb{P}(\hat{Z}_t^{x_0} \in dx) \leqslant \frac{c'}{\sqrt{t}} \exp\left(-\frac{[x - \vartheta_t^{x_0}]^2}{c't}\right), \tag{4.16}$$

where $\vartheta_t^{x_0}$ is the solution of the ODE:

$$\frac{d}{dt}\vartheta_t = b(\vartheta_t) + \alpha e'(t), \quad t \in [0,T], \tag{4.17}$$

with $\vartheta_0^{x_0} = x_0$. Above, the function $[0,T] \ni t \mapsto e(t)$ represents $[0,T] \ni t \mapsto \mathbb{E}(M_t)$ given $X_0 = x_0$, which means that the initial condition x_0 of X_0 upon which e depends is fixed once and for all, independently of the initial condition of ϑ. In particular, as the initial condition of ϑ varies, the function e does not. We emphasize that c' is independent of e and can be taken to be that defined in Proposition 4.5. Indeed, we can write $\mathbb{P}(\hat{Z}_t^{x_0} \in dx)$ as $\mathbb{P}(\hat{Z}_t^{x_0} - \vartheta_t^{x_0} \in d(x - \vartheta_t^{x_0}))$, with

$$\begin{gathered}
d\big(\hat{Z}_t^{x_0} - \vartheta_t^{x_0}\big) = F\big(t, \hat{Z}_t^{x_0} - \vartheta_t^{x_0}\big)dt + dW_t, \quad t \in [0,T] \; ; \quad \hat{Z}_0^{x_0} - \vartheta_0^{x_0} = 0 \; ; \\
F(t,x) = b\big(x + \vartheta_t^{x_0}\big) - b\big(\vartheta_t^{x_0}\big), \quad t \in [0,T], \quad x \in \mathbb{R}.
\end{gathered}$$

We then notice that $F(t,\cdot)$ is K-Lipschitz continuous (since b is) and satisfies $F(t,0) = 0$, so that, referring to [6], all the parameters involved in the definition of the constant c' are independent of e. The fact that c' is independent of e is crucial. As a consequence, we can bound $(d/dx)\mathbb{P}(\hat{Z}_{t-s}^{\sharp_s,0} \in dx)$ in a similar way, that is, with the same constant c' as in (4.16): for any $0 \leqslant s < t \leqslant T$, with $t - s \leqslant 1$,

$$\frac{d}{dx}\mathbb{P}(\hat{Z}_{t-s}^{\sharp_s,0} \in dx) \leqslant \frac{c'}{\sqrt{t-s}} \exp\left(-\frac{[x - \vartheta_{t-s}^{\sharp_s,0}]^2}{c'(t-s)}\right), \tag{4.18}$$

where $\vartheta^{\sharp_s,0}$ is the solution of the ODE:

$$\frac{d}{dt}\vartheta_t^{\sharp_s} = b\big(\vartheta_t^{\sharp_s}\big) + \alpha \frac{d}{dt} e^{\sharp_s}(t), \quad t \in [0, T-s],$$

with $\vartheta_0^{\sharp_s,0} = 0$ as initial condition.

Bound of the density in small time. Keep in mind that $X_0 = x_0 \leqslant 1-\epsilon$. Therefore, by the comparison principle for ODEs, $\vartheta_t^{x_0} \leqslant \vartheta_t^{1-\epsilon}$ for any $t \in [0,T]$, so that by Gronwall's Lemma

$$\vartheta_t^{x_0} \leqslant \vartheta_t^{1-\epsilon} \leqslant \big(1 - \epsilon + \Lambda T + \alpha e(T)\big)\exp(\Lambda T).$$

By Lemma 4.3, we know that $e(T) \leqslant B(T, \alpha, b)$, so that

$$\vartheta_t^{x_0} \leqslant \big(1 - \epsilon + \Lambda T + \alpha B(T, \alpha, b)\big)\exp(\Lambda T). \tag{4.19}$$

Now choose T_0 as in Proposition 4.5, i.e. $T_0 \leqslant 1$ such that

$$\big(1-\epsilon\big)\exp(\Lambda T_0) \leqslant 1 - 7\epsilon/8, \quad \Lambda T_0 \exp(\Lambda T_0) \leqslant \epsilon/8,$$

and then take $\alpha_1 \in (0,1)$ such that

$$\alpha_1 B(T_0, \alpha_1, b)\exp(\Lambda T_0) \leqslant \epsilon/4.$$

Then, whenever $\alpha \leqslant \alpha_1$, it holds that

$$\vartheta_t^{x_0} \leqslant 1 - \epsilon/2, \quad t \in [0, T_0 \wedge T].$$

Therefore, for $x \geqslant 1 - \epsilon/4$,

$$\exp\left(-\frac{[x - \vartheta_t^{x_0}]^2}{c't}\right) \leqslant \exp\left(-\frac{\epsilon^2}{16c't}\right), \quad t \in [0, T_0 \wedge T]. \tag{4.20}$$

Similarly,

$$\vartheta_{t-s}^{\sharp_s,0} \leqslant 3\epsilon/8 \leqslant 3/8, \quad 0 \leqslant s \leqslant t \leqslant T_0 \wedge T.$$

Indeed, $e^{\sharp_s}(T-s) \leqslant e(T)$ for $s \in [0,T]$, so that (4.19) applies to $\vartheta_{t-s}^{\sharp_s,0}$ with $1-\epsilon$ therein being replaced by 0. Therefore, for $x \geqslant 1-\epsilon/4$, it holds that $x - \vartheta_{t-s}^{\sharp_s,0} \geqslant 3/4 - 3/8 = 3/8 \geqslant 1/4$, so that

$$\exp\left(-\frac{[x - \vartheta_{t-s}^{\sharp_s,0}]^2}{c'(t-s)}\right) \leqslant \exp\left(-\frac{1}{16c'(t-s)}\right), \quad 0 \leqslant s < t \leqslant T_0 \wedge T. \tag{4.21}$$

In the end, for $x \in (1-\epsilon/4, 1)$ and $t \leqslant T_0 \wedge T$, we deduce from (4.15), (4.16), (4.18), (4.20), (4.21) and Lemma 4.3 again, that

$$\frac{d}{dx}\mathbb{P}(X_t \in dx) \leqslant c'\varpi_0\big[\epsilon^{-1} + e(T \wedge T_0)\big] \leqslant c'\varpi_0\big[\epsilon^{-1} + B(T_0, \alpha, b)\big], \tag{4.22}$$

where

$$\varpi_0 = \sup_{t>0}\left[t^{-1/2}\exp\left(-\frac{1}{16c't}\right)\right] = 4\sqrt{c'}\sup_{u>0}\big[u\exp(-u^2)\big] = 2^{3/2}\sqrt{c'}\exp\left(-\frac{1}{2}\right).$$

Bound of the density in long time. We now discuss what happens for $T > T_0$ and $t \in [T_0, T]$. Then,

$$\begin{aligned}\frac{d}{dx}\mathbb{P}(X_t \in dx) &\leqslant \frac{d}{dx}\mathbb{P}\left(X_t \in dx, \tau_1^{\sharp_{t-T_0}} \leqslant T_0\right) + \frac{d}{dx}\mathbb{P}\left(X_t \in dx, \tau_1^{\sharp_{t-T_0}} > T_0\right) \\ &= \pi_1 + \pi_2,\end{aligned} \tag{4.23}$$

with $\tau_1^{\sharp_{t-T_0}} = \inf\{u > 0 : X_{t-T_0+u-} \geqslant 1\} = \inf\{u > 0 : X_{u-}^{\sharp_{t-T_0}} \geqslant 1\}$. The above expression says that we split the event (X_t is in the neighbourhood of x) into two disjoint parts according to the fact that X reaches the threshold or not within the time window $[t - T_0, t]$. We have chosen this interval to be of length T_0 in order to apply the results in small time.

We first investigate π_2. The point is that, on the event that $\tau_1^{\sharp_{t-T_0}} > T_0$ and within the time window $[t - T_0, t]$, X behaves as a standard diffusion process without any jumps, namely as a process with the same dynamics as $\hat{Z}^{\sharp_{t-T_0}, X_{t-T_0}}$. Following (4.16), we then have

$$\begin{aligned}\pi_2 &= \frac{d}{dx}\mathbb{P}\left(\hat{Z}_{T_0}^{\sharp_{t-T_0}, X_{t-T_0}} \in dx, \tau_1^{\sharp_{t-T_0}} > T_0\right) \\ &\leqslant \frac{d}{dx}\mathbb{P}\left(\hat{Z}_{T_0}^{\sharp_{t-T_0}, X_{t-T_0}} \in dx\right) \leqslant \sup_{z \leqslant 1} \frac{d}{dx}\mathbb{P}\left(\hat{Z}_{T_0}^{\sharp_{t-T_0}, z} \in dx\right) \leqslant c' T_0^{-1/2}.\end{aligned} \tag{4.24}$$

We now turn to π_1. Here we write that

$$\begin{aligned}\pi_1 &= \frac{d}{dx}\mathbb{P}\left(X_t \in dx, \tau_1^{\sharp_{t-T_0}} \leqslant T_0\right) = \sum_{k \geqslant 1} \frac{d}{dx}\mathbb{P}\left(X_t \in dx, \tau_k^{\sharp_{t-T_0}} \leqslant T_0 < \tau_{k+1}^{\sharp_{t-T_0}}\right) \\ &= \sum_{k \geqslant 1} \int_0^{T_0} \frac{d}{dx}\mathbb{P}\left(X_t \in dx, \tau_k^{\sharp_{t-T_0}} \leqslant T_0 < \tau_{k+1}^{\sharp_{t-T_0}} | \tau_k^{\sharp_{t-T_0}} = s\right) \mathbb{P}(\tau_k^{\sharp_{t-T_0}} \in ds) \\ &= \sum_{k \geqslant 1} \int_0^{T_0} \frac{d}{dx}\mathbb{P}\left(\hat{Z}_{T_0-s}^{\sharp_{s+t-T_0}, 0} \in dx, T_0 < \tau_{k+1}^{\sharp_{t-T_0}}\right) \mathbb{P}(\tau_k^{\sharp_{t-T_0}} \in ds),\end{aligned}$$

since on the event that $\tau_k^{\sharp_{t-T_0}} \leqslant T_0 < \tau_{k+1}^{\sharp_{t-T_0}}$ and given that the k-th (and last) jump of X in the interval $[t - T_0, t]$ occurs at time $t - T_0 + s$ with $s \in [0, T_0]$, we have that the process X_r for $r \in [t - T_0 + s, t]$ coincides with the process $\hat{Z}_u^{\sharp_{s+t-T_0}, 0}$ for $u \in [0, T_0 - s]$. Thus

$$\begin{aligned}\pi_1 &\leqslant \sum_{k \geqslant 1} \int_0^{T_0} \frac{d}{dx}\mathbb{P}\left(\hat{Z}_{T_0-s}^{\sharp_{s+t-T_0}, 0} \in dx\right) \mathbb{P}(\tau_k^{\sharp_{t-T_0}} \in ds) \\ &= \int_0^{T_0} \frac{d}{dx}\mathbb{P}\left(\hat{Z}_{T_0-s}^{\sharp_{s+t-T_0}, 0} \in dx\right) e'(s + t - T_0) ds.\end{aligned} \tag{4.25}$$

By (4.18), we have

$$\begin{aligned}
&\int_0^{T_0} \frac{d}{dx}\mathbb{P}\left(\hat{Z}_{T_0-s}^{\sharp_{s+t-T_0},0} \in dx\right) e'(s+t-T_0)ds \\
&\quad \leqslant \int_0^{T_0} \frac{c'}{\sqrt{T_0-s}} \exp\left(-\frac{[x-\vartheta_{T_0-s}^{\sharp_{s+t-T_0},0}]^2}{c'(T_0-s)}\right) e'(s+t-T_0)ds.
\end{aligned}$$

Following the shift in (4.12), it is well seen that the mapping $[0,T_0] \ni s \mapsto e^{\sharp_{t-T_0}}(s)$ satisfies Lemma 4.3, that is

$$\sup_{0 \leqslant s \leqslant T_0} e^{\sharp_{t-T_0}}(s) = \sup_{0 \leqslant s \leqslant T_0} \big[e(s+t-T_0) - e(t-T_0)\big] = e(t) - e(t-T_0) \leqslant B(T_0,\alpha,b).$$

Therefore, we can follow the same strategy as in short time, see (4.21) and (4.22). Indeed, for $\alpha \leqslant \alpha_1$, by the choice of T_0 as before, it holds that

$$\pi_1 \leqslant c'\varpi_0 B(T_0,\alpha,b),$$

for $x \in [1-\epsilon/4,1)$. Using (4.24) and the above bound, we deduce that, for $t \in [T_0,T]$,

$$\frac{d}{dx}\mathbb{P}\left(X_t \in dx\right) \leqslant c'\big[T_0^{-1/2} + \varpi_0 B(T_0,\alpha,b)\big].$$

□

Now, we can complete the proof of Proposition 4.5.

Proof of Proposition 4.5. Proposition 4.5 follows from the combination of Lemmas 4.4 and 4.6. Indeed, given T_0 and α_0 as defined in Proposition 4.5, then by Lemma 4.6 it follows that $\mathbb{P}(X_t \in A) < (1/\alpha)|A|$ for any Borel subset $A \subset [1-\epsilon/4,1]$, any $\alpha < \alpha_0$ and any $t \in [0,T]$. The result follows by Lemma 4.4, with $\mathcal{B}$ being given by $\mathcal{B}_0 \exp(2\Lambda)$ with ϵ in $\mathcal{B}_0$ replaced by $\epsilon/4$. □

4.4. **Estimate of the density of the killed process.** In light of the previous subsection, for a solution $(X_t)_{0 \leqslant t \leqslant T}$ to (2.1) such that the mapping $[0,T] \ni t \mapsto e(t) = \mathbb{E}(M_t)$ is continuously differentiable, we here investigate

$$\frac{d}{dx}\mathbb{P}\left(X_t \in dx, t < \tau_1\right), \quad t \in [0,T],\ x \leqslant 1,$$

where $\tau_1 = \inf\{t > 0 : X_{t-} \geqslant 1\}$ as usual. This is the density of the killed process $(X_{t\wedge\tau_1})_{0 \leqslant t \leqslant T}$, which makes sense because of Lemmas 3.3 and 4.6.

Here is the main result of this subsection:

Lemma 4.7. *For any $\epsilon \in (0,1)$, any $T > 0$ and any $\mathcal{B} > 0$, there exist two constants μ_T and η_T, only depending upon T, $\mathcal{B}$, ϵ, K and Λ, such that, for any initial condition $x_0 \leqslant 1-\epsilon$ and any continuously differentiable non-decreasing deterministic mapping $[0,T] \ni t \mapsto e(t)$ satisfying*

$$e(0) = 0 \quad ; \quad e(t) - e(s) \leqslant \mathcal{B}(t-s)^{1/2}, \quad 0 \leqslant s \leqslant t \leqslant T,$$

if $(\chi_t)_{0 \leqslant t \leqslant T}$ denotes the solution of the SDE

$$d\chi_t = b(\chi_t)dt + \alpha e'(t)dt + dW_t, \quad t \in [0,T] \ ; \quad \chi_0 = x_0,$$

then, with $p(t,y)$ denoting the density of χ_t killed at 1 as in (3.3), we have

$$p(t,y) \leqslant \mu_T (1-y)^{\eta_T}, \quad t \in [0,T], \ y \in [1-\epsilon/4, 1]. \tag{4.26}$$

Proof. First Step. The first step is to provide a probabilistic representation for p. For a given $(T,x) \in (0,+\infty) \times (-\infty, 1)$, we consider the solution to the SDE:

$$dY_t = -\big[b(Y_t) + \alpha e'(T-t)\big] dt + dW_t, \quad t \in [0,T] \ ; \quad Y_0 = y, \tag{4.27}$$

together with some stopping time $\rho \leqslant \rho_0 \wedge T$, where $\rho_0 = \inf\{t \in [0,T] : Y_t \geqslant 1\}$ (with $\inf \emptyset = +\infty$). Then, by Lemma 3.3 and the Itô-Krylov formula (see [12, Chapter II, Section 10]),

$$\begin{aligned} & d\big(p(T-t,Y_t)\big) \\ & = -\partial_t p(T-t,Y_t)dt - \big[b(Y_t) + \alpha e'(T-t)\big]\partial_y p(T-t,Y_t)dt + \frac{1}{2}\partial^2_{yy} p(T-t,Y_t)dt \\ & \quad + \partial_y p(T-t,Y_t)dW_t \\ & = b'(Y_t)p(T-t,Y_t)dt + \partial_y p(T-t,Y_t)dW_t, \end{aligned}$$

for $0 \leqslant t \leqslant \rho$. Therefore, the Feynman-Kac formula yields

$$p(T,y) = \mathbb{E}\bigg[p(T-\rho,Y_\rho)\mathbf{1}_{\{Y_\rho \neq 1\}} \exp\bigg(-\int_0^\rho b'(Y_s)ds\bigg)\bigg], \tag{4.28}$$

the indicator function following from the Dirichlet boundary condition satisfied by $p(\cdot, 1)$.

Second Step. We now specify the choice of ρ. Given some free parameters $L \geqslant 1$ and $\delta \in (0, \epsilon/4)$ such that $L\delta \leqslant \epsilon/4$, we assume that the initial condition y in (4.27) is in $(1-\delta, 1)$ and then consider the stopping time

$$\rho = \inf\{t \in [0,T] : Y_t \notin (1-L\delta, 1)\} \wedge \delta^2. \tag{4.29}$$

Assume that $\delta^2 \leqslant T$. By (4.28), we deduce that

$$p(T,y) \leqslant \exp(K\delta^2)(1 - \mathbb{P}(Y_\rho = 1)) \sup_{(t,z)\in\mathcal{Q}(\delta,L)} p(t,z), \tag{4.30}$$

with

$$\mathcal{Q}(\delta,L) = \big\{(t,z) \in [T-\delta^2, T] \times [1-L\delta, 1]\big\}.$$

The point is then to give a lower bound for $\mathbb{P}(Y_\rho = 1)$. By assumption, we know that e is $\mathcal{B}$-Hölder continuous on $[0,T]$. Therefore, since $Y_0 = y \in (1-\delta, 1)$, we have, for any $t \in [0,\rho]$,

$$Y_t \geqslant 1 - \delta - m\delta^2 - \alpha\mathcal{B}\delta + W_t,$$

with

$$m = \sup_{0 \leqslant z \leqslant 1} |b(z)|. \tag{4.31}$$

Therefore, for $m\delta \leqslant 1$,

$$Y_t \geqslant 1 - 2\delta - \alpha\mathcal{B}\delta + W_t, \quad t \in [0,\rho],$$

so that

$$\big\{ Y_\rho = 1 \big\} \supset \left\{ \sup_{0 \leqslant t \leqslant \delta^2} W_t > (2 + \alpha \mathcal{B})\delta \right\} \cap \left\{ \inf_{0 \leqslant t \leqslant \delta^2} W_t > (2 + \alpha \mathcal{B} - L)\delta \right\}. \quad (4.32)$$

Choosing $L = 3 + \alpha \mathcal{B}$ and applying a scaling argument, we deduce that

$$\begin{aligned} &\mathbb{P}\left(\left\{ \sup_{0 \leqslant t \leqslant \delta^2} W_t > (2 + \alpha \mathcal{B})\delta \right\} \cap \left\{ \inf_{0 \leqslant t \leqslant \delta^2} W_t > (2 + \alpha \mathcal{B} - L)\delta \right\} \right) \\ &= \mathbb{P}\left(\left\{ \sup_{0 \leqslant t \leqslant 1} W_t > (2 + \alpha \mathcal{B}) \right\} \cap \left\{ \inf_{0 \leqslant t \leqslant 1} W_t > -1 \right\} \right) =: c'' \in (0,1). \end{aligned} \quad (4.33)$$

We note that the above quantity c'' is independent of δ and T. Moreover, we deduce from (4.32) that $\mathbb{P}(Y_\rho = 1) \geqslant c''$ and therefore, from (4.30), that

$$p(T, y) \leqslant (1 - c'') \exp(K\delta^2) \sup_{z \in \mathcal{I}(L\delta)} \sup_{t \in [0,T]} p(t, z),$$

with $\mathcal{I}(r) = [1 - r, 1]$, for $r > 0$. Choosing δ small enough such that $(1 - c'') \exp(K\delta^2) \leqslant (1 - c''/2)$, we obtain

$$p(T, y) \leqslant \left(1 - \frac{c''}{2}\right) \sup_{z \in \mathcal{I}(L\delta)} \sup_{t \in [0,T]} p(t, z), \quad y \in \mathcal{I}(\delta).$$

Modifying c'' if necessary (c'' being chosen as small as needed), we can summarize the above inequality as follows: for $\delta \leqslant c''$,

$$p(T, y) \leqslant (1 - c'') \sup_{z \in \mathcal{I}(L\delta)} \sup_{t \in [0,T]} p(t, z), \quad y \in \mathcal{I}(\delta). \quad (4.34)$$

We now look at what happens when $T \leqslant \delta^2$ in (4.30). In such a case ρ in the previous argument must be replaced by $\rho \wedge T$. Observing that $p(T - \rho \wedge T, Y_{\rho \wedge T}) = 0$ on the event $\{\rho \geqslant T\} \cup \{Y_{\rho \wedge T} = 1\}$ (since $p(0, \cdot) = 0$ on $[1 - \epsilon/4, 1]$) and following (4.30), we obtain, for $y \in \mathcal{I}(\delta)$,

$$p(T, y) \leqslant \exp(K\delta^2) \big[1 - \mathbb{P}\big(\{ Y_{\rho \wedge T} = 1 \} \cup \{ \rho \geqslant T \} \big) \big] \sup_{(t,z) \in \mathcal{Q}'(\delta, L)} p(t, z), \quad (4.35)$$

with $\mathcal{Q}'(\delta, L) = \{(t, z) \in [0, T] \times [1 - L\delta, 1]\}$. Now, the right-hand side of (4.32) is included in the event $\{Y_{\rho \wedge T} = 1\} \cup \{\rho \geqslant T\}$ so that (4.33) yields a lower bound for $\mathbb{P}(\{Y_{\rho \wedge T} = 1\} \cup \{\rho \geqslant T\})$. Therefore, we can repeat the previous arguments in order to prove that (4.34) also holds when $T \leqslant \delta^2$, which means that (4.34) holds true in both cases.

Therefore, by replacing T by t in the left-hand side in (4.34) and by letting t vary within $[0, T]$, we have in any case,

$$\sup_{y \in \mathcal{I}(\delta)} \sup_{t \in [0,T]} p(t, y) \leqslant (1 - c'') \sup_{z \in \mathcal{I}(L\delta)} \sup_{t \in [0,T]} p(t, z).$$

By induction, for any integer $n \geqslant 1$ such that $L^n \delta \leqslant r_0$, with $r_0 = c'' \wedge (\epsilon/4)$,

$$\sup_{y \in \mathcal{I}(\delta)} \sup_{t \in [0,T]} p(t, y) \leqslant (1 - c'')^n \sup_{z \in \mathcal{I}(L^n\delta)} \sup_{t \in [0,T]} p(t, z).$$

Given $\delta \in (0, r_0/L)$, the maximal value for n is $n = \lfloor \ln[r_0/\delta]/\ln L \rfloor$. We deduce that, for any $\delta \in (0, r_0/L)$,

$$\sup_{y \in \mathcal{I}(\delta)} \sup_{t \in [0,T]} p(t,y) \leqslant (1-c'')^{(\ln[r_0/\delta]/\ln L)-1} \sup_{z \in \mathcal{I}(\epsilon/4)} \sup_{t \in [0,T]} p(t,z). \tag{4.36}$$

Following (4.16), we know that

$$\sup_{z \in \mathcal{I}(\epsilon/4)} \sup_{t \in [0,T]} p(t,z) \leqslant \sup_{z \in \mathcal{I}(\epsilon/4)} \sup_{t \in [0,T]} \left[\frac{c_T}{\sqrt{t}} \exp\left(-\frac{[z - \vartheta_t^{x_0}]^2}{c_T t} \right) \right], \tag{4.37}$$

for some constant c_T only depending upon T and K and where $(\vartheta_t^{x_0})_{0 \leqslant t \leqslant T}$ stands for the solution of the ODE

$$\frac{d\vartheta}{dt} = b(\vartheta_t) + \alpha e'(t), \quad t \in [0,T] \; ; \quad \vartheta_0 = x_0.$$

Pay attention that we here use the same notation as in (4.17) for the solution of the above ODE but here $e(t)$ is not given as some $\mathbb{E}(M_t)$. Actually, we feel that there is no possible confusion here. Notice also that e is fixed and does not depend upon the initial condition x_0.

By the comparison principle for ODEs and then by Gronwall's Lemma, we deduce from the fact that e is $\mathcal{B}$-Hölder continuous of exponent $1/2$ that

$$\vartheta_t^{x_0} \leqslant \vartheta_t^{1-\epsilon} \leqslant \left[1 - \epsilon + \Lambda t + \mathcal{B} t^{1/2}\right] \exp(\Lambda t), \quad t \in [0,T].$$

Using the above inequality, we can bound the right-hand side in (4.37). Precisely, the above inequality says that the exponential term in the supremum decays exponentially fast as t tends to 0 so that the term inside the supremum can be bounded when t is small; when t is bounded away from 0, the term inside the supremum is bounded by $c_T/\sqrt{t}$. It is plain to deduce that

$$\sup_{z \in \mathcal{I}(\epsilon/4)} \sup_{t \in [0,T]} p(t,z) \leqslant c_T, \tag{4.38}$$

for a new value of c_T, possibly depending on ϵ as well. Therefore, for $\delta \in (0, r_0/L)$, (4.36) yields

$$\sup_{y \in \mathcal{I}(\delta)} \sup_{t \in [0,T]} p(t,y) \leqslant \frac{c_T}{(1-c'')} \left(\frac{\delta}{r_0} \right)^{\eta},$$

with $\eta = -\ln(1-c'')/\ln L$. This proves (4.26) for $y \in (1 - r_0/L, 1)$. Note that η is here independent of T, contrary to what is indicated in the statement of Lemma 4.7. However, we feel it is simpler to indicate T in η_T as the constant $\mathcal{B}$ in the sequel will be chosen in terms of T thus making η depend on T. Using (4.38), we can easily extend the bound to any $y \in (1 - \epsilon/4, 1)$ by modifying if necessary the parameters μ_T and η_T therein. This completes the proof. □

4.5. **Bound for the gradient.** Here is the final step to complete the proof of Theorem 4.1:

Proposition 4.8. *For any $\epsilon \in (0,1)$, any $\mathcal{B} > 0$ and any $T > 0$, there exists a constant $\mathcal{M}_T > 0$, only depending upon T, $\mathcal{B}$, ϵ, K and Λ, such that, for any initial*

condition $x_0 < 1 - \epsilon$, any continuously differentiable non-decreasing deterministic mapping $[0,T] \ni t \mapsto e(t)$ satisfying

$$e(0) = 0 \quad ; \quad e(t) - e(s) \leqslant \mathcal{B}(t-s)^{1/2}, \quad 0 \leqslant s \leqslant t \leqslant T,$$

if $(\chi_t)_{0 \leqslant t \leqslant T}$ denotes the solution of the SDE

$$d\chi_t = b(\chi_t)dt + \alpha e'(t) + dW_t, \quad t \in [0,T] \ ; \quad \chi_0 = x_0,$$

then, for any integer n such that $n \geqslant \lceil 4/\epsilon \rceil$,

$$|\partial_y p(t,1)| \leqslant \frac{\mathcal{M}_T n^{-\eta_T}}{1 - \exp[-\mathcal{M}_T^{-1}(1+\alpha C_T)n^{-1}]}(1+\alpha C_T), \quad t \in [0,T],$$

where $p(t,y)$ is the density of χ_t killed at 1 as in (3.3)*, η_T is as in Lemma 4.7, and*

$$C_T = \sup_{0 \leqslant t \leqslant T} e'(t).$$

Proof. We consider the barrier function

$$q(t,y) = \Theta \exp(Kt)\big[1 - \exp(\gamma(y-1))\big], \quad t \geqslant 0, \ y \in \mathbb{R}, \tag{4.39}$$

where γ and Θ are free nonnegative parameters. Then, for $t > 0$ and $y < 1$,

$$\begin{aligned}
&\partial_t q(t,y) + \big(b(y) + \alpha e'(t)\big)\partial_y q(t,y) - \frac{1}{2}\partial^2_{yy} q(t,y) \\
&= \Theta \exp(Kt)\exp(\gamma(y-1))\big(-(b(y) + \alpha e'(t))\gamma + \frac{1}{2}\gamma^2\big) + Kq(t,y).
\end{aligned}$$

Keeping in mind that

$$\sup_{0 \leqslant t \leqslant T} e'(t) = C_T,$$

and choosing

$$\gamma = 2(\max(m,1) + \alpha C_T), \tag{4.40}$$

where $m = \sup_{0 \leqslant z \leqslant 1} |b(z)|$ as before, we obtain, for $t \in [0,T]$ and $y \in (0,1)$,

$$-\big(b(y) + \alpha e'(t)\big)\gamma + \frac{1}{2}\gamma^2 \geqslant -2(\max(m,1) + \alpha C_T)^2 + 2(\max(m,1) + \alpha C_T)^2 = 0.$$

Thus, for $t \in [0,T]$ and $y \in (0,1)$,

$$\partial_t q(t,y) + \big(b(y) + \alpha e'(t)\big)\partial_y q(t,y) - \frac{1}{2}\partial^2_{yy} q(t,y) \geqslant Kq(t,y) \geqslant -b'(y)q(t,y),$$

which reads

$$\partial_t q(t,y) + \partial_y\big[\big(b(y) + \alpha e'(t)\big)q(t,y)\big] - \frac{1}{2}\partial^2_{yy} q(t,y) \geqslant 0. \tag{4.41}$$

For a given integer $n \geqslant \lceil 4/\epsilon \rceil$, we choose Θ as the solution of

$$\Theta\left[1 - \exp\left(-\frac{2(\max(m,1) + \alpha C_T)}{n}\right)\right] = \mu_T n^{-\eta_T}, \tag{4.42}$$

with μ_T and η_T as in the statement of Lemma 4.7. Pay attention that the factor in the left-hand side cannot be 0 as $\max(m,1)>0$. Notice also q thus depends upon n. By Lemma 4.7, we deduce that

$$q\left(t,1-\frac{1}{n}\right) \geqslant p\left(t,1-\frac{1}{n}\right), \quad 0 \leqslant t \leqslant T.$$

Now, we can apply the comparison principle for PDEs. Indeed, we also observe that $q(0,y) \geqslant p(0,y)=0$ for $y \in [1-1/n,1]$ and $q(t,1)=p(t,1)=0$ for $t \in [0,T]$. Therefore, by (4.41), we have

$$p(t,y) \leqslant q(t,y), \quad t \in [0,T],\ y \in \left[1-\frac{1}{n},1\right]. \tag{4.43}$$

Since $p(t,1)=0=q(t,1)$, we deduce

$$|\partial_y p(t,1)| \leqslant |\partial_y q(t,1)| = \frac{2\mu_T(\max(m,1)+\alpha C_T)n^{-\eta_T}}{1-\exp[-2(\max(m,1)+\alpha C_T)/n]}\exp(Kt). \tag{4.44}$$

□

We now complete the proof of Theorem 4.1. We make use of Proposition 3.4. Recall (3.6)

$$e'(t) = -\int_0^t \frac{1}{2}\partial_y p^{(0,s)}(t-s,1)e'(s)ds - \frac{1}{2}\partial_y p(t,1), \quad t \in [0,T],$$

where p represents the density of the process X killed at 1 and $p^{(0,s)}$ represents the density of the process $X^{\sharp_s}$ driven by $e^{\sharp_s}=e(\cdot+s)-e(s)$ (see (4.12)) killed at 1 with $X_0^{\sharp_s}=0$ as initial condition.

By Proposition 4.5 and Lemma 4.3, we know that, for a given $s \in [0,T)$ and for the prescribed values of α, the mapping $[0,T-s] \ni r \mapsto e^{\sharp_s}(r)$ is 1/2-Hölder continuous, the Hölder constant only depending upon T, α, ϵ, K and Λ (Proposition 4.5 permits to bound the increments of $e^{\sharp_s}$ on small intervals and Lemma 4.3 gives a trivial bound for the increments of $e^{\sharp_s}$ on large intervals). Therefore, by Proposition 4.8, we know that

$$|\partial_y p^{(0,s)}(t-s,1)| \leqslant \frac{\mathcal{M}_T n^{-\eta_T}}{1-\exp[-\mathcal{M}_T^{-1}(1+\alpha C_T)n^{-1}]}(1+\alpha C_T), \quad t \in [s,T], \tag{4.45}$$

for $n \geqslant \lceil 4/\epsilon \rceil$ and for some constant $\mathcal{M}_T$ only depending upon T, α, ϵ, K and Λ. The same bound also holds true for $\partial_y p(t,1)$.

We deduce that, for any $t \in [0,T]$ and any n such that $n \geqslant \lceil 4/\epsilon \rceil$,

$$e'(t) \leqslant \frac{\mathcal{M}_T n^{-\eta_T}}{1-\exp[-\mathcal{M}_T^{-1}(1+\alpha C_T)n^{-1}]}(1+\alpha C_T)\frac{e(T)+1}{2}.$$

By Lemma 4.3, we have a bound for $e(T)=\mathbb{E}(M_T)$, which means that we can bound $(e(T)+1)/2$ in the right-hand side above by modifying the constant $\mathcal{M}_T$. Recalling

$$C_T = \sup_{0 \leqslant t \leqslant T} e'(t),$$

we deduce that

$$C_T\big(1-\exp[-\mathcal{M}_T^{-1}(1+\alpha C_T)n^{-1}]\big) \leqslant \mathcal{M}_T(1+\alpha C_T)n^{-\eta_T}. \tag{4.46}$$

Choosing n large enough such that the right hand side is less than $(1+\alpha C_T)/2$ (so that n depends on T) and multiplying by α, we get (since $\alpha \in (0,1)$):

$$\frac{\alpha C_T}{2} \leqslant \frac{1}{2} + \alpha C_T \exp[-\mathcal{M}_T^{-1}(1+\alpha C_T)n^{-1}].$$

This shows that αC_T must be bounded in terms of $\mathcal{M}_T$ and n. Precisely, we have

$$\alpha C_T \leqslant 1 + 2\sup_{r\geqslant 0}\big[r\exp[-\mathcal{M}_T^{-1}(1+r)n^{-1}]\big] := R < +\infty.$$

By (4.46), we deduce that

$$C_T \leqslant \sup_{0\leqslant r\leqslant R}\left[\frac{\mathcal{M}_T(1+r)n^{-\eta_T}}{1-\exp[-\mathcal{M}_T^{-1}(1+r)n^{-1}]}\right],$$

which is independent of α (for $\alpha \in (0,\alpha_0]$), as required. ∎

5. Existence and uniqueness for all time

We now put everything together to arrive at our goal, which is Theorem 5.2 below.

Lemma 5.1. *For any $T>0$, initial condition $X_0 = x_0 < 1$, and $\alpha < \alpha_0$, where $\alpha_0 = \alpha_0(x_0)$ is as in Theorem 4.1, there exists a constant $C_{\mathrm{den}}(T)$ depending only on T, x_0, K and Λ such that any solution to* (2.1) *on $[0,T]$ according to Definition 2.3 satisfies*

$$\frac{d}{dy}\mathbb{P}\left(X_t \in dy\right) \leqslant C_{\mathrm{den}}(T)(1-y),$$

for all $y \in (1-\epsilon/8, 1)$ and $t\in[0,T]$, with $\epsilon = \min(1, 1-x_0)$.

Proof. We assume that $(X_t)_{0\leqslant t\leqslant T}$ is the unique solution to (2.1) with $X_0 = x_0$ up until time T, and set $e(t) = \mathbb{E}(M_t)$. Following the notation of Section 3 (see also the last part of the proof of Theorem 4.1), for $y \leqslant 1$ and $t \leqslant T$, let

$$p(t,y) := \frac{d}{dy}\mathbb{P}\left(X_t \in dy, t<\tau_1\right),$$

$$p^{(0,s)}(t,y) := \frac{d}{dy}\mathbb{P}\left(X_t^{\sharp_s} \in dy, t < \tau_1^{\sharp_s} | X_0^{\sharp_s} = 0\right).$$

By Theorem 4.1, we know that e is $\mathcal{M}_T$-Lipschitz continuous, so that by (4.43),

$$p(t,y) \leqslant q(t,y), \quad t\in[0,T],\ y \in \left[1-\frac{1}{n}, 1\right],$$

where n stands for $\lceil 4/\epsilon\rceil$ and q is given by (4.39), with γ and Θ being fixed by (4.40) and (4.42), with $C_T = \mathcal{M}_T$. By the specific form of q, this says that there exists a constant C_T', depending only on T, x_0, K and Λ, such that

$$p(t,y) \leqslant C_T'(1-y), \quad t\in[0,T],\ y\in\left[1-\frac{\epsilon}{8}, 1\right],$$

using the elementary inequality $1 - \exp(-x) \leqslant x$ for $x \in \mathbb{R}$. Clearly, the same argument applies to $p^{(0,s)}(t-s,y)$, i.e.

$$p^{(0,s)}(t-s,y) \leqslant C'_T(1-y), \quad 0 \leqslant s < t \leqslant T,\ y \in \left[1 - \frac{\epsilon}{8}, 1\right].$$

Now, following the proof of (4.11), we get for $t \in [0,T]$ and $y \in [1-\epsilon/8, 1]$,

$$\begin{aligned} \frac{d}{dy}\mathbb{P}(X_t \in dy) &= p(t,y) + \int_0^t p^{(0,s)}(t-s,y)e'(s)ds \\ &\leqslant C'_T(1+e(T))(1-y), \end{aligned} \tag{5.1}$$

where we use Lemma 3.3 for justifying the passage to the density in (4.11). By Lemma 4.3, this completes the proof. □

Theorem 5.2. *For any $T > 0$, initial condition $X_0 = x_0 < 1$, and $\alpha < \alpha_0$, where $\alpha_0 = \alpha_0(x_0)$ is as in Theorem 4.1, there exists an unique solution to the nonlinear equation* (2.1) *on* $[0,T]$ *according to Definition 2.3.*

Proof. We would like a solution up until fixed time $T > 0$. The idea is to iterate Theorem 3.1, which is possible thanks to Lemma 5.1. Indeed, by Theorem 3.1, we have that there exists a solution to (2.1) with $X_0 = x_0$ up until some small time $T_1 > 0$. By Lemma 5.1, we therefore have that

$$\frac{d}{dy}\mathbb{P}(X_{T_1} \in dy) \leqslant C_{\mathrm{den}}(T_1)(1-y), \quad y \in \left[1 - \frac{\epsilon}{8}, 1\right],$$

where $\epsilon = \min(1-x_0, 1)$. If $T_1 \geqslant T$ we are done. If not, we have the above density bound for $(d/dy)\mathbb{P}(X_{T_1} \in dy)$ and we have an explicit bound for $\mathbb{E}|X_{T_1}|$ deriving from Lemma 4.3 and from Gronwall's Lemma. We also know from (5.1) and Lemma 3.3 that the density of X_{T_1} is differentiable at $y = 1$. Therefore, we can apply Theorem 3.1 again to see that there exists a solution to (2.1) on some interval $[T_1, T_1 + T_2]$ starting from X_{T_1}. As T_2 depends upon X_{T_1} through ϵ, $C_{\mathrm{den}}(T_1)$ and $\mathbb{E}|X_{T_1}|$ only and as these quantities can be bounded in terms of T, ϵ, K, Λ only, we then see that

$$T_2 \geqslant \phi(T)$$

for some constant $\phi(T)$ that refers to T, α, ϵ, K, Λ only. Now we know that there exists a solution to (2.1) with $X_0 = x_0$ on $[0, T_1 + T_2]$. If $T_1 + T_2 > T$ we are done. If not, by Lemma 5.1 once again,

$$\frac{d}{dy}\mathbb{P}(X_{T_1+T_2} \in dy) \leqslant C_{\mathrm{den}}(T_1+T_2)(1-y), \quad y \in \left[1 - \frac{\epsilon}{8}, 1\right],$$

and we can then repeat the argument n times to get a solution up until time $T_1 + \cdots + T_n$, where all $T_k \geqslant \phi(T)$ for $k \geqslant 2$ i.e. each time step is of size at least $\phi(T)$. It is then clear that there exists $n \geqslant 1$ such that $T_1 + \cdots + T_n \geqslant T$, and so we are done for the existence of a solution.

Uniqueness of the solution proceeds in the same way. Given another solution $(X'_t, M'_t)_{0 \leqslant t \leqslant T}$ on the interval $[0,T]$ in the sense of Definition 2.3, it must satisfy the *a priori* estimates in the statements of Theorem 4.1 and Lemmas 4.3 and 5.1. In particular, dividing the interval $[0,T]$ into subintervals of length $\phi(T)$ (except for

the last interval the length of which might be less than $\phi(T)$), with the same $\phi(T)$ as above, we can apply the contraction property in Theorem 3.1 on each subinterval iteratively. Precisely, choosing A_1 accordingly in Theorem 3.1, we prove by induction that the two solutions coincide on $[0, \phi(T)]$, $[0, 2\phi(T)]$, and so on. □

Remark 5.3 (A closed formula for $e(t) = \mathbb{E}(M_t)$)**.** *This remark is motivated by the proof of finite time blow-up in* [3] *(Theorem 2.2). In fact, thanks to the knowledge about the existence of a solution for α sufficiently small provided by Theorem 5.2, we can extract some interesting formulae by modifying the methods in* [3].

Indeed, fix $x_0 < 1$ and suppose that we are in the Ornstein-Uhlenbeck case, so that $b(x) = -\lambda x$ with $\lambda \geqslant 0$. Let $\alpha_0 = \alpha_0(x_0)$ be as in Theorem 5.2. Then for $\alpha < \alpha_0$ a solution to (2.1) *with $X_0 = x_0 < 1$ exists for all time in the sense of Definition 2.3. Set $e(t) = \mathbb{E}(M_t)$.*

For $\mu > 0$, let $\varphi(x) = \exp(\mu x)$. Then since $e(t)$ is continuously differentiable, by the generalized Itô formula we see that for all $t \geqslant 0$,

$$\begin{aligned}\mathbb{E}[\varphi(\exp(\lambda t)X_t)] = \mathbb{E}(\varphi(X_0)) &+ \int_0^t \alpha\mu \exp(\lambda s)e'(s)\mathbb{E}[\varphi(\exp(\lambda s)X_s)]ds \\ &+ \frac{\mu^2}{2}\int_0^t \exp(2\lambda s)\mathbb{E}[\varphi(\exp(\lambda s)X_s)]ds \\ &+ \int_0^t [\varphi(0) - \varphi(\exp(\lambda s))]e'(s)ds.\end{aligned}$$

Differentiating with respect to t and writing $z(t) = \mathbb{E}[\varphi(\exp(\lambda t)X_t)]$ yields

$$\frac{d}{dt}z(t) = \left[\alpha\mu\exp(\lambda t)e'(t) + \frac{\mu^2}{2}\exp(2\lambda t)\right] z(t) + [\varphi(0) - \varphi(\exp(\lambda t))]e'(t)$$

for all $t > 0$, with initial condition $z(0) = \exp(\mu x_0)$. Writing $\eta(t) := \alpha\mu\exp(\lambda t)e'(t) + (\mu^2/2)\exp(2\lambda t)$ and $\nu(t) := [\varphi(\exp(\lambda t)) - \varphi(0)]e'(t) \geqslant 0$, we can solve this first-order ordinary differential equation in the standard way to see that

$$z(t) = \exp\left(\int_0^t \eta(s)ds\right)\left[z(0) - \int_0^t \nu(s)\exp\left(-\int_0^s \eta(u)du\right)ds\right].$$

Now, by the definition of φ, we can note that $z(0) \geqslant 0$, and that

$$z(t)\exp(-\mu\exp(\lambda t)) \leqslant 1, \qquad t \geqslant 0,$$

since $X_t < 1$. Setting

$$\gamma(t) := \exp\left(-\mu\exp(\lambda t) + \int_0^t \eta(s)ds\right),$$

it follows that

$$0 \leqslant \gamma(t)\left[z(0) - \int_0^t \nu(s)\exp\left(-\int_0^s \eta(u)du\right)ds\right] \leqslant 1, \tag{5.2}$$

for all $t \geqslant 0$. But we can see that $\gamma(t) \to \infty$ as $t \to \infty$ for all $\mu > 0$. Indeed,

$$\begin{aligned}
\gamma(t) &= \exp\left(-\mu \exp(\lambda t) + \int_0^t \left[\alpha\mu \exp(\lambda s) e'(s) + (\mu^2/2)\exp(2\lambda s)\right] ds\right) \\
&\geqslant \exp\left(-\mu \exp(\lambda t) + \frac{\mu^2}{2}\int_0^t \exp(2\lambda s) ds\right) \\
&= \left\{ \begin{array}{ll} \exp\left(\mu \exp(\lambda t)\left[\frac{\mu}{4\lambda}\exp(\lambda t) - 1\right] - \frac{\mu^2}{4\lambda}\right) & \text{if } \lambda > 0 \\ \exp\left(-\mu + \frac{\mu^2}{2} t\right) & \text{if } \lambda = 0 \end{array} \right\} \to \infty.
\end{aligned}$$

Thus for the bound (5.2) to remain true when $t \to \infty$, we must have

$$z(0) - \int_0^\infty \nu(s) \exp\left(-\int_0^s \eta(u) du\right) ds = 0, \qquad \forall \mu > 0,$$

which provides a surprising closed formula for e'. We can then perform an integration by parts. Indeed, by writing $\hat{e}(s) = \int_0^s \exp(\lambda u) e'(u) du$ for all $s \geqslant 0$, the above equality yields

$$\begin{aligned}
0 &= z(0) - \int_0^\infty [\varphi(\exp(\lambda s)) - \varphi(0)] e'(s) \exp\left(-\alpha\mu\hat{e}(s) - \frac{\mu^2}{4\lambda}[\exp(2\lambda s) - 1]\right) ds \\
&= z(0) - \int_0^\infty \hat{\nu}(s) \alpha\mu \hat{e}'(s) \exp\left(-\alpha\mu\hat{e}(s)\right) ds,
\end{aligned}$$

with the convention that $[\exp(2\lambda s) - 1]/(4\lambda) = s/2$ if $\lambda = 0$, and where

$$\hat{\nu}(s) = \frac{\varphi(\exp(\lambda s)) - \varphi(0)}{\alpha\mu} \exp\left(-\frac{\mu^2}{4\lambda}[\exp(2\lambda s) - 1]\right) \exp(-\lambda s).$$

Since it is clear that $\hat{\nu}(\infty) = 0$, we can perform an integration by parts, so that

$$0 = z(0) - \hat{\nu}(0) - \int_0^\infty \hat{\nu}'(s) \exp\left(-\alpha\mu\hat{e}(s)\right) ds.$$

In other words, whenever $\alpha < \alpha_0$, we must have that the map $t \mapsto e(t) = \mathbb{E}(M_t)$ satisfies

$$\exp(\mu x_0) = \frac{e^\mu - 1}{\alpha\mu} + \int_0^\infty \hat{\nu}'(s) \exp\left(-\alpha\mu\hat{e}(s)\right) ds, \tag{5.3}$$

for all $\mu > 0$. It seems to be an interesting question to determine whether this characterizes e and more generally to determine whether the existence of a solution to (5.3) implies the existence of a global solution to (2.1). In particular, in the case when $\lambda = 0$, (5.3) reads

$$\frac{\mu^2}{2}\int_0^\infty \exp\left(-\alpha\mu e(s) - \frac{\mu^2}{2} s\right) ds = 1 - \frac{\alpha\mu \exp(\mu x_0)}{\exp(\mu) - 1}. \tag{5.4}$$

6. Explicit bounds for the criticality

Theorem 5.2 says that, for a given initial condition x_0, existence and uniqueness of a solution to (2.1) holds when the self-excitation rate α lies in $(0, \alpha_0(x_0))$, for some $\alpha_0(x_0) \in (0, 1]$, the value of which is given in the statements of both Theorem 4.1 and Proposition 4.5. This raises several questions:

(1) Does the model indeed exhibit several regimes for global solvability according to the values of α?
(2) If so, are there two phases exactly (using an analogy with statistical mechanics)? To put it differently, does there exist a critical threshold α_c such that global solvability holds for $\alpha < \alpha_c$ and fails for $\alpha > \alpha_c$? Or are there any more complex cases when global solvability fails for some α^+ and then holds for some $\alpha^- > \alpha^+$?
(3) Conversely, are there any cases when global solvability holds for any $\alpha < 1$?

Although Theorem 5.2 does not provide any answers to these questions, it does say that, for a given initial condition $x_0 < 1$, the critical rate

$$\alpha_c^-(x_0) = \sup\bigl\{\alpha^- \in (0,1] \ : \ \mathcal{G}(\alpha, x_0) \text{ holds for any } \alpha \in (0, \alpha^-)\bigr\}$$

is (strictly) positive, since it is larger than $\alpha_0(x_0)$, where $\mathcal{G}(\alpha, x_0)$ stands for the property of global unique solvability of equation (2.1) when driven by α and initialized with x_0.

In this framework, the lower bound $\alpha_c^-(x_0) \geqslant \alpha_0(x_0)$, with $\alpha_0(x_0)$ given by Proposition 4.5 and Theorem 4.1, is complementary to the previous result in [3]: there it is indeed proven that the existence of a global solution fails for some values of $\alpha < 1$ and some initial conditions, including Dirac masses located right below the barrier 1, in the case when $b(x) + \lambda x \geqslant 0$, with $\lambda > 0$. This shows that, in some cases, $\alpha_c^-(x_0)$ is indeed strictly less than 1. Actually, the result in [3] says more: for any value of $\lambda > 0$, there exists a critical threshold $x_c < 1$ such that, for any $x_0 \in (x_c, 1)$, there exists some $\alpha_0'(x_0) < 1$ such that global existence fails (in the sense that e' blows up in finite time) for any $\alpha \in (\alpha_0'(x_0), 1)$. In other words, the critical rate

$$\alpha_c^+(x_0) = \inf\bigl\{\alpha^+ \in (0,1] \ : \ \mathcal{G}(\alpha, x_0) \text{ fails for any } \alpha \in (\alpha^+, 1)\bigr\}$$

is strictly less than 1 for x_0 close to 1, with the convention that $\alpha_c^+(x_0) = 1$ if the set in the right-hand side is empty. Obviously,

$$\alpha_c^-(x_0) \leqslant \alpha_c^+(x_0),$$

but, as written above, the following questions remain open:

(1) Are there any cases when $\alpha_c^-(x_0) \neq \alpha_c^+(x_0)$?
(2) Are there any cases when $\alpha_c^-(x_0) = 1$?

More generally, finding the optimal values of $\alpha_c^-(x_0)$ or/and $\alpha_c^+(x_0)$ turns out to be a really challenging problem.

In this section, we address two problems: (i) the first is to provide an explicit lower bound for $\alpha_c^-(0)$ in the case when $b(x) = -\lambda x$, $x \in \mathbb{R}$, for some $\lambda > 0$; (ii) the second is to provide a lower bound for $\alpha_c^-(x_0)$ and an upper bound for $\alpha_c^+(x_0)$

in the case when $b = 0$ but x_0 is chosen in such a way that $\alpha_c^+(x_0)$ is known to be less than 1 and then to discuss the bounds in light of some numerical experiments.

6.1. **Ornstein-Uhlenbeck case.** In this paragraph, we investigate further the case when $b(x) = -\lambda x$ (referred to as the Ornstein-Uhlenbeck case, since the dynamics of X up until the first jump time coincide with those of the Ornstein-Uhlenbeck process), which is the choice most often made in neuroscience. As above, we assume that the diffusion coefficient is 1. We then exhibit a lower bound for the optimal value of α_c^- according to the methodology described in Section 4. As the method involves rather tedious computations, we will focus on the case $X_0 = 0$ only: this makes things a bit simpler. We thus consider

$$X_t = -\lambda \int_0^t X_s ds + \alpha e(t) + W_t - M_t, \quad t \geqslant 0, \tag{6.1}$$

where $e(t) = \mathbb{E}(M_t)$. When $X_0 = 0$, as is the case here, numerical experiments indicate that existence and uniqueness of a global solution may hold for any $\alpha \in (0, 1)$. Indeed, equation (6.1) is then solved by using a particle method: we refer to the forthcoming paper [5] for a complete description and analysis of the numerical scheme. For instance, when $\lambda = 0.01$, we get as a plot of the mapping $t \mapsto e(t)$ associated to the approximate solution that given in Figure 1.

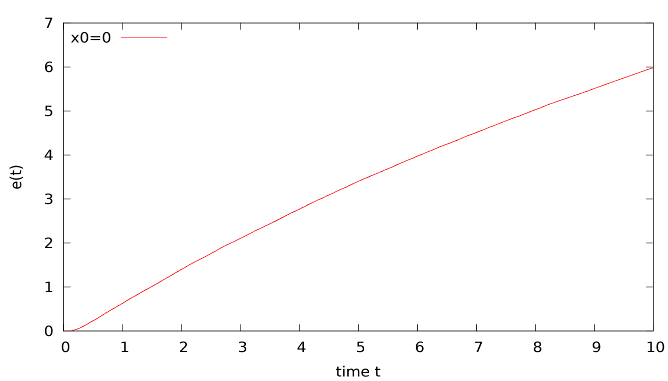

FIGURE 1. Plot of $t \mapsto e(t)$ for $X_0 = 0$, $\lambda = 0.01$, $\alpha = 0.9$ and $t \in [0, 10]$.

Numerically speaking, existence and uniqueness is then suspected to hold since the graph of the mapping $t \mapsto e(t)$ is smooth and, in particular, has no jump. We refer to Subsection 6.2 for a counter-example when a jump occurs for some $x_0 \neq 0$, indicating that existence of a global solution, as defined above, cannot hold. Below, we are not able to prove that $\alpha_c^-(0) = 1$, showing that our methodology is not optimal. Anyhow, we manage to give an explicit lower bound for the critical value of $\alpha_c^-(0)$ and to prove that it converges toward 1 when λ tends to $+\infty$.

The strategy is to seek α_0 such that, for any $t \geqslant 0$ and y in a neighbourhood of 1, $[d/dy]\mathbb{P}(X_t \in dy) < 1/\alpha$ for all $\alpha < \alpha_0$, so that the condition (4.5) in Lemma 4.4

is satisfied (which then allows us to pass to an unique global solution thanks to the results of Sections 4 and 5).

First Step. The first step is to provide some *a priori* estimates for the solution on any interval $[0,T]$ where existence holds. Precisely, the point is to refine the *a priori* L^∞ bound for the function $(e(t) = \mathbb{E}(M_t))_{0 \leqslant t \leqslant T}$ given in the statement of Lemma 4.3, by taking advantage of the mean-reverting property of the Ornstein-Uhlenbeck process. As in Section 4, in all the *a priori* estimates below we assume that $(X_t, M_t)_{0 \leqslant t \leqslant T}$ stands for a solution to (6.1) in the sense of Definition 2.3.

Here is the first claim:

Lemma 6.1. *For any* $t \in [0,T]$,

$$\max\left(\mathbb{E}\big[[(X_t)_-]^2\big]^{1/2},\ \mathbb{E}\big[(X_t)_+\big],\ (1-\alpha)\exp(-\lambda t)\int_0^t \exp(\lambda s)e'(s)ds\right) \leqslant f_1(\lambda,t),$$
$$\text{with} \quad f_1(\lambda,t) = \left[\frac{1-\exp(-2\lambda t)}{2\lambda}\right]^{1/2},$$

where $(x)_- = \max(-x,0)$ *for* $x \in \mathbb{R}$.

Proof. We first investigate how far the process $(X_t)_{0 \leqslant t \leqslant T}$ can go in the negative direction. The point is to apply Itô's formula to $([(X_t)_-]^2)_{0 \leqslant t \leqslant T}$:

$$\begin{aligned}
&[(X_t)_-]^2 \\
&= 2\lambda \int_0^t (X_{s-})_- X_s ds - 2\alpha \int_0^t (X_{s-})_- e'(s)ds + \int_0^t \mathbf{1}_{\{X_{s-}<0\}} ds - 2\int_0^t (X_{s-})_- dW_s \\
&= -2\lambda \int_0^t (X_s)_-^2 ds + 2\alpha \int_0^t X_s \mathbf{1}_{\{X_s<0\}} e'(s)ds + \int_0^t \mathbf{1}_{\{X_s<0\}} ds - 2\int_0^t (X_{s-})_- dW_s,
\end{aligned}$$

for $t \in [0,T]$. Here, $(X_{s-})_-$ stands for the negative part of the left-limit of X at time s. To pass from the second to the third line, notice that $X_{s-} = X_s$ when $X_{s-} \leqslant 0$. Multiplying by $\exp(2\lambda t)$, reapplying Itô's formula and then taking the expectation, we deduce:

$$\mathbb{E}\big[\exp(2\lambda t)(X_t)_-^2\big] \leqslant \int_0^t \exp(2\lambda s)ds = \frac{\exp(2\lambda t)-1}{2\lambda} = \exp(2\lambda t) f_1^2(\lambda,t). \tag{6.2}$$

Now, multiplying in the same way (6.1) by $\exp(\lambda t)$ and applying Itô's formula, we get:

$$\exp(\lambda t)X_t = \alpha \int_0^t \exp(\lambda s)e'(s)ds + \int_0^t \exp(\lambda s)dW_s - \int_0^t \exp(\lambda s)dM_s.$$

Taking the expectation, we deduce:

$$(1-\alpha)\int_0^t \exp(\lambda s)e'(s)ds = -\mathbb{E}\big[\exp(\lambda t)X_t\big],$$

so that, writing $X_t = (X_t)_+ - (X_t)_-$, with $(x)_+ = \max(x,0)$,

$$\begin{aligned}\mathbb{E}\big[\exp(\lambda t)(X_t)_+\big] + (1-\alpha)\int_0^t \exp(\lambda s)e'(s)ds &= \mathbb{E}\big[\exp(\lambda t)(X_t)_-\big] \\ &\leqslant \exp(\lambda t) f_1(\lambda,t),\end{aligned} \tag{6.3}$$

where the last inequality follows from (6.2) and Hölder's inequality. □

We deduce:

Lemma 6.2. *For any $t,t' \in [0,T]$, $t \leqslant t'$*

$$e(t') - e(t) \leqslant \frac{f_2(\lambda, t'-t, T)}{1-\alpha}$$

with

$$f_2(\lambda,r,T) = \min\big[\exp(1) f_1(\lambda,T)\big(1+\lambda r\big), 1 + (\lambda/2)^{1/2} r + (2r/\pi)^{1/2}\big].$$

Proof. We prove first that the bound holds with respect to the first argument in the minimum defining f_2. By Lemma 6.1, we know that, for $0 \leqslant t \leqslant t+h \leqslant T$,

$$[e(t+h) - e(t)]\exp(-\lambda h) \leqslant \exp[-\lambda(t+h)] \int_0^{t+h} \exp(\lambda s) e'(s) ds \leqslant \frac{f_1(\lambda,T)}{1-\alpha}.$$

Choosing $h = 1/\lambda$, we deduce that for $0 \leqslant t \leqslant s \leqslant T \wedge (t + 1/\lambda)$

$$e(s) - e(t) \leqslant \exp(1)\frac{f_1(\lambda,T)}{1-\alpha}.$$

Summing over a subdivision of $[t,t']$ made of small intervals of lengths less than $1/\lambda$, we complete the proof for the first argument in the minimum.

We now turn to the second argument in the minimum. As in the proof of Lemma 4.4, we can assume without any loss of generality that $t=0$ by shifting the system (the shift being of length t), provided the new initial condition X_0 is chosen as an arbitrary random variable. Precisely, the new initial condition X_0 is then understood as the previous X_t. Similarly, the new variables $(X_s)_{0 \leqslant s \leqslant t'-t}$ are understood as the previous $(X_s)_{t \leqslant s \leqslant t'}$ so that, by Lemma 6.1, we can assume the new $(X_s)_{0 \leqslant s \leqslant t'-t}$ satisfy

$$\mathbb{E}\big[(X_s)_+\big] \leqslant f_1(\lambda,T) \leqslant \left[\frac{1}{2\lambda}\right]^{1/2}.$$

Following the proof of (2.10)–(2.11), we then get for any $s \in [0, t'-t]$:

$$\begin{aligned}e(s) = \mathbb{E}(M_s) \leqslant \mathbb{E}\left[\sup_{0 \leqslant r \leqslant s} (Z_r)_+\right] &\leqslant 1 + \lambda \int_0^s \mathbb{E}\big[(X_s)_+\big] ds + \alpha e(s) + \mathbb{E}\left[\sup_{0 \leqslant r \leqslant s} W_r\right] \\ &\leqslant 1 + \left[\frac{\lambda}{2}\right]^{1/2}(t'-t) + \alpha e(s) + \mathbb{E}\left[\sup_{0 \leqslant r \leqslant s} W_r\right],\end{aligned}$$

so that, for any $s \in [0, t'-t]$,

$$e(s) \leqslant \frac{1 + (\lambda/2)^{1/2}(t'-t) + \sqrt{2(t'-t)/\pi}}{1-\alpha},$$

the last term in the right-hand side following from

$$\mathbb{E}\left(\sup_{0\leqslant s\leqslant t} W_s\right) = \sqrt{\frac{2t}{\pi}}. \tag{6.4}$$

Indeed, by the reflection principle,

$$\mathbb{P}\left(\sup_{s\leqslant t} W_s \leqslant a\right) = 2\mathbb{P}\left(W_t \leqslant a\right) - 1 = \sqrt{\frac{2}{t\pi}}\int_0^a e^{-z^2/2t}dz,$$

so that

$$\mathbb{E}\left(\sup_{s\leqslant t} W_s\right) = \sqrt{\frac{2}{t\pi}}\int_0^\infty ze^{-z^2/2t}dz = \sqrt{\frac{2t}{\pi}}.$$

□

Second Step. According to the program we fixed in the proof of Lemma 4.6, we now investigate the bound of the density in small time. In the Ornstein-Uhlenbeck case, (4.16) (with $x_0 = 0$) has the form:

$$\begin{aligned}\frac{d}{dx}\mathbb{P}\left(\hat{Z}_t^0 \in dx\right) &= \frac{1}{(2\pi)^{1/2} f_1(\lambda, t)} \\ &\times \exp\left[-\frac{1}{2f_1^2(\lambda,t)}\left(x - \alpha\exp(-\lambda t)\int_0^t \exp(\lambda r)e'(r)dr\right)^2\right],\end{aligned} \tag{6.5}$$

for $t \leqslant T$, with $f_1(\lambda, t)$ as in the statement of Lemma 6.1 and $\hat{Z}_t^0$ given by (4.14) . More generally, (4.18) reads:

$$\begin{aligned}\frac{d}{dx}\mathbb{P}\left(\hat{Z}_{t-s}^{\sharp_s,0} \in dx\right) &= \frac{1}{(2\pi)^{1/2} f_1(\lambda, t-s)} \\ \times\ \exp&\left[-\frac{1}{2f_1^2(\lambda,t-s)}\left(x - \alpha\exp[-\lambda(t-s)]\int_0^{t-s} \exp(\lambda r)e'(s+r)dr\right)^2\right],\end{aligned}$$

for $s < t \leqslant T$. By Lemma 6.1, we notice that the "mean-term" in the exponential is bounded by

$$\begin{aligned}\alpha\exp\big[-\lambda(t-s)\big]\int_0^{t-s}\exp(\lambda r)e'(s+r)dr &= \alpha\exp(-\lambda t)\int_s^t \exp(\lambda r)e'(r)dr \\ &\leqslant \frac{\alpha f_1(\lambda,t)}{1-\alpha}.\end{aligned} \tag{6.6}$$

By Lemma 6.2, it is also bounded by

$$\alpha\exp\big[-\lambda(t-s)\big]\int_0^{t-s}\exp(\lambda r)e'(s+r)dr \leqslant \alpha(e(t)-e(s)) \leqslant \frac{\alpha f_2(\lambda, t-s, t)}{1-\alpha}. \tag{6.7}$$

Therefore, since $r \mapsto f_2(\lambda, r, t)$ is non-decreasing, if, for some $\varepsilon \in (0, 1)$,

$$\frac{\alpha}{1-\alpha}\min\big[f_1(\lambda,t), f_2(\lambda,t,t)\big] \leqslant 1-\varepsilon,$$

then, for $x \in (1-\varepsilon\delta, 1)$, with $\delta \in (0,1)$, we have for any $s \in [0,t)$ (pay attention that this covers (6.5) as well):

$$\begin{aligned}\frac{d}{dx}\mathbb{P}\left(\hat{Z}_{t-s}^{\sharp_s,0} \in dx\right) &\leqslant \frac{1}{(2\pi)^{1/2} f_1(\lambda, t-s)} \exp\left[-\frac{(1-\delta)^2\varepsilon^2}{2f_1^2(\lambda, t-s)}\right] \\ &\leqslant \frac{1}{(1-\delta)\varepsilon} g\left(\frac{(1-\delta)\varepsilon}{f_1(\lambda,t)}\right),\end{aligned}$$

where

$$g(u) = \sup_{v>u}\left[\frac{v}{(2\pi)^{1/2}}\exp\left(-\frac{v^2}{2}\right)\right] = \begin{cases} \dfrac{u}{(2\pi)^{1/2}}\exp\left(-\dfrac{u^2}{2}\right) & \text{if } u \geqslant 1, \\ \dfrac{1}{(2\pi)^{1/2}}\exp\left(-\dfrac{1}{2}\right) & \text{if } u < 1, \end{cases}$$

using the fact that $u \mapsto g(u)$ is non-increasing and $r \mapsto f_1(\lambda, r)$ is non-decreasing. Therefore, by (4.15), for $x \in (1-\varepsilon\delta, 1)$

$$\frac{d}{dx}\mathbb{P}(X_t \in dx) \leqslant \frac{1+e(t)}{(1-\delta)\varepsilon} g\left(\frac{(1-\delta)\varepsilon}{f_1(\lambda,t)}\right).$$

Define $T_0 = 1 \wedge \lambda^{-1}$. Then, by Lemma 6.2, for $t \leqslant T \wedge T_0$,

$$\frac{d}{dx}\mathbb{P}(X_t \in dx) \leqslant \left(1 + \frac{f_2(\lambda, T_0, T_0)}{1-\alpha}\right)\frac{1}{(1-\delta)\varepsilon} g\left(\frac{(1-\delta)\varepsilon}{f_1(\lambda, T_0)}\right) \tag{6.8}$$

again since g is non-increasing, provided

$$\frac{\alpha}{1-\alpha}\min\left[f_1(\lambda, T_0), f_2(\lambda, T_0, T_0)\right] \leqslant 1-\varepsilon \tag{6.9}$$

holds true.

Third Step. We now investigate the bound of the density for $t \geqslant T_0$. We first discuss π_1 defined by (4.23). By (4.25) (plugging inside the current value of T_0), and the same arguments as in the second step,

$$\pi_1 \leqslant \frac{1}{(1-\delta)\varepsilon} g\left(\frac{(1-\delta)\varepsilon}{f_1(\lambda, T_0)}\right)(e(t) - e(t-T_0)),$$

provided

$$\frac{\alpha}{1-\alpha}\min\left[f_1(\lambda, \infty), f_2(\lambda, T_0, \infty)\right] \leqslant 1-\varepsilon. \tag{6.10}$$

Pay attention that, in $f_1(\lambda, \cdot)$ and $f_2(\lambda, T_0, \cdot)$, $\cdot$ is understood as T_0 in (6.9) whereas it is understood as ∞ in (6.10). The difference between them comes from the fact that in this step t may be large. By Lemma 6.2,

$$\pi_1 \leqslant \frac{1}{(1-\delta)\varepsilon} g\left(\frac{(1-\delta)\varepsilon}{f_1(\lambda, T_0)}\right)\frac{f_2(\lambda, T_0, \infty)}{1-\alpha}. \tag{6.11}$$

It now remains to investigate π_2 also defined in (4.23). We start with the case $\lambda \geqslant 1$, so that $T_0 = \lambda^{-1}$. Following (4.24) and (6.5) (keeping in mind that $T_0\lambda = 1$), we have

$$\pi_2 \leqslant \sup_{z \leqslant 1} \pi_2(z), \tag{6.12}$$

with

$$\pi_2(z) = \frac{1}{(2\pi)^{1/2} f_1(\lambda, 1/\lambda)} \times \exp\left[-\frac{1}{2f_1^2(\lambda, 1/\lambda)}\left(x - \exp(-1)\left[z + \alpha \int_0^{1/\lambda} \exp(\lambda r) e'(t - T_0 + r) dr\right]\right)^2\right].$$

Actually, the bound can be refined for λ large by taking into account the fact that z is understood as a realization of X_{t-T_0} and that the probability that X_{t-T_0} is away from 0 is very small when λ is large because of the mean-reverting property of the Ornstein-Uhlenbeck process. Precisely, we claim:

$$\begin{aligned}\pi_2 &\leqslant \max\left(\sup_{z \leqslant 1/2} \pi_2(z), \sup_{z \leqslant 1} \pi_2(z) \mathbb{P}(X_{t-T_0} \geqslant 1/2)\right) \\ &\leqslant \max\left(\sup_{z \leqslant 1/2} \pi_2(z), \left(2/\lambda\right)^{1/2} \sup_{z \leqslant 1} \pi_2(z)\right),\end{aligned} \tag{6.13}$$

the last line following from Lemma 6.1 and Markov's inequality. As above (see (6.6) and (6.7)), we know that the mean term in the exponential in $\pi_2(z)$ is bounded by

$$\begin{aligned}&\exp(-1)z + \alpha \exp(-1) \int_0^{1/\lambda} \exp(\lambda r) e'(t - T_0 + r) dr \\ &= \exp(-1)z + \alpha \exp(-\lambda T_0) \int_0^{T_0} \exp(\lambda r) e'(t - T_0 + r) dr \\ &= \exp(-1)z + \alpha \exp(-\lambda t) \int_{t-T_0}^{t} \exp(\lambda r) e'(r) dr \\ &\leqslant \exp(-1)z + \frac{\alpha}{1-\alpha} \min\left[f_1(\lambda, \infty), f_2(\lambda, T_0, \infty)\right].\end{aligned}$$

Therefore, if (6.10) holds for $\varepsilon \in (\exp(-1), 1)$, then, for $x \in (1 - (\varepsilon - \exp(-1))\delta, 1)$, we get from (6.12) and (6.13):

$$\pi_2 \leqslant \min\left[f_4(\delta, \lambda), f_5(\delta, \lambda)\right],$$

with

$$\begin{aligned}f_4(\delta, \lambda) &= \frac{1}{(2\pi)^{1/2} f_1(\lambda, 1/\lambda)} \exp\left[-\frac{(1-\delta)^2(\varepsilon - \exp(-1))^2}{2f_1^2(\lambda, 1/\lambda)}\right], \\ f_5(\delta, \lambda) &= \max\left\{\frac{1}{(2\pi)^{1/2} f_1(\lambda, 1/\lambda)} \exp\left[-\frac{(1-\delta)^2(\varepsilon - \exp(-1)/2)^2}{2f_1^2(\lambda, 1/\lambda)}\right], \frac{2^{1/2} f_4(\delta, \lambda)}{\lambda^{1/2}}\right\}.\end{aligned}$$

Finally, we have

$$\begin{aligned}\frac{d}{dx}\mathbb{P}\left(X_t \in dx\right) \leqslant &\frac{1}{(1-\delta)\varepsilon} g\left(\frac{(1-\delta)\varepsilon}{f_1(\lambda, T_0)}\right) \frac{f_2(\lambda, T_0, \infty)}{1-\alpha} \\ &+ \min\left[f_4(\delta, \lambda), f_5(\delta, \lambda)\right].\end{aligned} \tag{6.14}$$

provided (6.10) holds true, since $[d/dx]\mathbb{P}(X_t \in dx) \leqslant \pi_1 + \pi_2$ by (4.24).

We now turn to the case $\lambda < 1$. Then, $T_0 = 1$ and

$$\pi_2 \leqslant \frac{1}{(2\pi)^{1/2} f_1(\lambda, 1)}.$$

Therefore,

$$\frac{d}{dx}\mathbb{P}\big(X_t \in dx\big) \leqslant \frac{1}{(1-\delta)\varepsilon} g\left(\frac{(1-\delta)\varepsilon}{f_1(\lambda, T_0)}\right)\frac{f_2(\lambda, T_0, \infty)}{1-\alpha} + \frac{1}{(2\pi)^{1/2} f_1(\lambda, 1)}. \tag{6.15}$$

Conclusion. For our purpose, it is worth noting that δ can be chosen as small as desired, so that the critical value $\alpha_c^-(0)$ can be obtained by putting $\delta = 0$ in all the previous work. The reason is quite simple: the strategy for proving Proposition 4.5 applies provided the estimate of the density (4.5) holds true on a small interval of the form $(1-\delta, 1)$, δ being possibly really small. Obviously, when δ tends to 0, the resulting Hölder constant in Proposition 4.5 blows up, but this does not affect the determination of $\alpha_c^-(0)$.

Start with the case $\lambda \geqslant 1$. From (6.8), (6.10) (which implies (6.9)) and (6.14), we get as critical conditions for α in order to satisfy (4.5):

$$\begin{aligned}
&\alpha\left[1 + \frac{\min\big[f_1(\lambda,\infty), f_2(\lambda, T_0, \infty)\big]}{1-\varepsilon}\right] < 1,\\
&\frac{1}{\varepsilon} g\left(\frac{\varepsilon}{f_1(\lambda, T_0)}\right)\alpha + \frac{1}{\varepsilon} f_2(\lambda, T_0, T_0) g\left(\frac{\varepsilon}{f_1(\lambda, T_0)}\right)\frac{\alpha}{1-\alpha} < 1,\\
&\min\big[f_4(\delta,\lambda), f_5(\delta,\lambda)\big]\alpha + \frac{1}{\varepsilon} f_2(\lambda, T_0, \infty) g\left(\frac{\varepsilon}{f_1(\lambda, T_0)}\right)\frac{\alpha}{1-\alpha} < 1.
\end{aligned} \tag{6.16}$$

When $\lambda < 1$, we get similar conditions as in (6.16), but the third line in (6.16) is replaced by

$$\frac{1}{(2\pi)^{1/2} f_1(\lambda, T_0)}\alpha + \frac{1}{\varepsilon} f_2(\lambda, T_0, \infty) g\left(\frac{\varepsilon}{f_1(\lambda, T_0)}\right)\frac{\alpha}{1-\alpha} < 1. \tag{6.17}$$

The first inequality in (6.16) is explicit. The two last (and (6.17) as well) are of the form

$$\begin{aligned}
c_1\alpha + c_2\frac{\alpha}{1-\alpha} < 1 &\Leftrightarrow c_1\alpha^2 - (1 + c_1 + c_2)\alpha + 1 > 0\\
&\Leftrightarrow \alpha < \frac{1 + c_1 + c_2 - [(1+c_1+c_2)^2 - 4c_1]^{1/2}}{2c_1},
\end{aligned}$$

where we used $\alpha \in (0,1)$ in the two equivalences. Therefore, a lower bound for the critical value of α can be computed numerically: we denote it by α_0^ε as the coefficients in (6.16) depend upon $\varepsilon \in (\exp(-1), 1)$. Optimizing α_0^ε over ε, we get a new bound, which must be read as the α_0 (or $\alpha_0(0)$ if indicating the initial condition) in Proposition 4.5 and in Theorem 4.1. See Figure 2 for a plot of $\alpha_0(0)$ in terms of λ (the optimization of α_0^ε with respect to ε being performed numerically over a mesh of step size 0.01 for the parameter ε) and Figure 3 for a zoom on the initial values.

The minimum is $\alpha_0^{\min}(0) \approx 0.301$. It is reached at $\lambda \approx 0.23$, which is the point at which Figure 3 has a singular behaviour. Actually, such a singularity is expected to

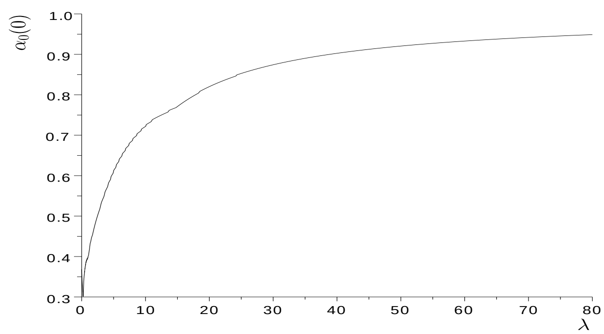

FIGURE 2. Plot of $\alpha_0(0)$ in terms of $\lambda \in [0; 80]$.

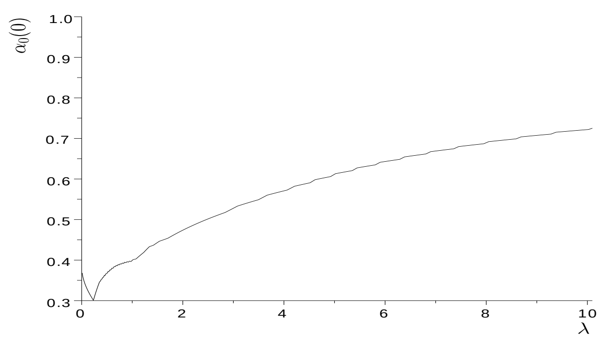

FIGURE 3. Plot of $\alpha_0(0)$ in terms of $\lambda \in [0; 10]$.

be of computational essence and certainly indicates that the computations we have performed are far from being optimal, especially for λ small. Anyhow, for λ large, Figure 2 shows that the critical value $\alpha_c^-(0)$ is (at worse) close to 1, which sounds as an interesting result from a practical point of view. Indeed, by scaling, the case $\lambda \gg 1$ and $\sigma = 1$ (with σ as in (1.2)) is equivalent to the case $\lambda = 1$ and $\sigma \ll 1$: for a macroscopic value of λ, the system is thus uniquely solvable for most of the values of α in $(0, 1)$ when the noise is small. Actually, the convergence of $\alpha_0(0)$ toward 1 as $\lambda \to \infty$ can be proven rigorously, as stated in Proposition 6.3 below.

Proposition 6.3. *In the case $X_0 = 0$ and $b(x) = -\lambda x$ (with $\sigma = 1$), the critical value α_0 in Theorem 4.1 and Proposition 4.5 can be chosen such that, for λ large,*

$$\begin{aligned}\alpha_0 = \left[1 + \frac{1}{(1-\exp(-1))(2\lambda)^{1/2}}\right]^{-1} &\sim_{\lambda\to+\infty} 1 - \frac{1}{(1-\exp(-1))(2\lambda)^{1/2}}.\\ &\approx 1 - 1.12\lambda^{-1/2}.\end{aligned}$$

In particular, $\alpha_0 \to 1$ as $\lambda \to +\infty$.

Proof. The proof is quite straightforward and follows from the fact that, in (6.16), when ε is given, the coefficients of α and $\alpha/(1-\alpha)$ in the second and third lines

tend to 0 as λ tends to the infinity, so that the conditions are automatically satisfied for λ large. In the first line, the condition is of the form

$$\alpha \left[1 + \frac{1}{(1-\varepsilon)(2\lambda)^{1/2}} \right] < 1.$$

Letting ε tend to the minimal authorized value $\exp(-1)$, we complete the proof. $\square$

The idea of the proof of Proposition 6.3 is thus very simple: the optimal strategy when λ is large consists in choosing ε close to the minimal authorized value $\exp(-1)$. Such a strategy is well-checked from a numerical point of view. Indeed, Figure 4 provides a plot of the optimal choice for ε in terms of λ, when optimizing α_0^ε with respect to ε (ε being chosen amongst a mesh of step size 0.01):

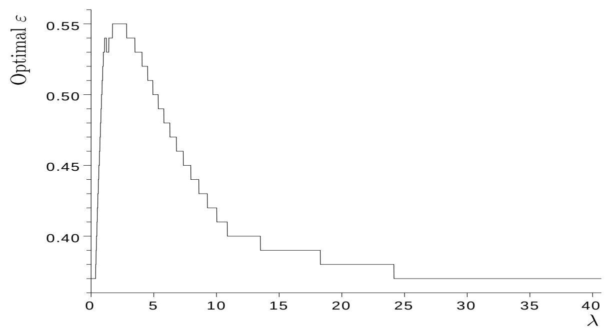

FIGURE 4. Plot of the optimal value of ε in terms of $\lambda \in [0; 40]$.

6.2. **A case with blow-up.** We now investigate the Brownian case (so that $b \equiv 0$). We have already proven that, for any initial condition $X_0 = x_0 < 1$, $\alpha_c^-(x_0) > 0$. On the other hand, by adapting the strategy in [3], we can prove that $\alpha_c^+(x_0) < 1$ for some x_0 (note that the result in [3] is written for a non-zero λ).

Indeed, from Remark 5.3, we know that, for any solution $(X_t, M_t)_{t \geqslant 0}$ to (2.1) with $b \equiv 0$, (5.4) must be satisfied. If

$$\exp(\mu x_0) \geqslant \frac{\exp(\mu) - 1}{\alpha\mu}, \tag{6.18}$$

then (5.4) implies that the integral $\int_0^\infty \exp(-\alpha\mu e(s) - \mu^2 s/2) ds$ must be non-positive, which is clearly a contradiction. In such cases, existence of a solution must fail.

Minimizing over μ in (6.18), we deduce that existence fails for $\alpha > \inf_{\mu>0}[[\exp(\mu) - 1]/[\mu \exp(\mu x_0)]]$, so that $\alpha_c^+(x_0) \leqslant \inf_{\mu>0}[[\exp(\mu) - 1]/[\mu \exp(\mu x_0)]]$ (if less than 1, which might not be the case). As in the previous subsection, this value must be compared with numerical experiments. Indeed, by investigating numerically the graphs of $e(t) = \mathbb{E}(M_t)$ for different values of α, we can detect the emergence of some discontinuity and thus give a bound for $\alpha_c^+(x_0)$.

Choosing $x_0 = 0.8$, we find $\inf_{\mu>0}[[\exp(\mu) - 1]/[\mu \exp(0.8\mu)]] \approx 0.539$, so that $\alpha_c^+(0.8) \leqslant 0.54$. In Figure 5, we numerically observe that the graph of e is regular for $\alpha = 0.38$ but has a jump for $\alpha = 0.39$. We also observe that the associated particle system keeps stable for $\alpha = 0.38$ but blows up for $\alpha = 0.39$. From the observations we have for other values of α, it seems that, in this framework, global solvability fails for $\alpha \geqslant 0.39$ and holds for $\alpha \leqslant 0.38$. We thus expect that, in this framework, $\alpha_c^+(0.8)$ and $\alpha_c^-(0.8)$ coincide, with a common value between 0.38 and 0.39. Figure 6 below summarizes these facts (when $b(x) \equiv 0$ and $x_0 = 0.8$). For α in region **D**, by the above argument, we know that global uniqueness fails. For α in region **C**, by numerical experiments it seems that global uniqueness also fails, but we are unable to prove it. Similarly, we have proven in this article that for α in region **A**, for some $\alpha_0(0.8) > 0$, an unique solution for all time exists, while numerically it seems that global solutions also exist for α in region **B**. We thus suspect that $\alpha_c^-(0.8) = \alpha_c^+(0.8)$ lies in the region in between **B** and **C**.

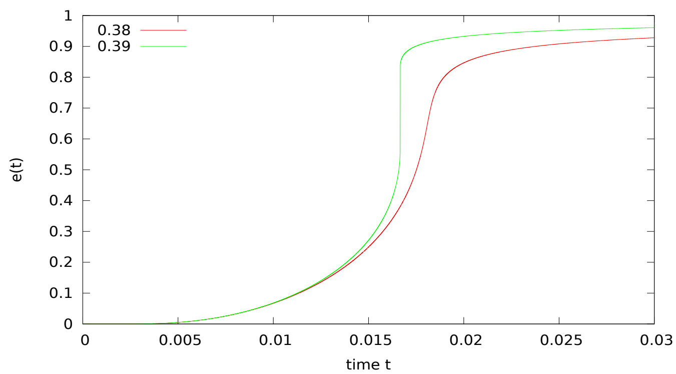

FIGURE 5. Plot of $t \mapsto e(t)$ for $x_0 = 0.8$, $b(x) \equiv 0$, $\alpha = 0.38$ (red) and $\alpha = 0.39$ (green).

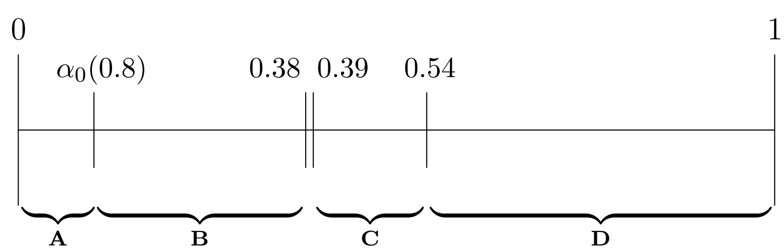

FIGURE 6. Critical regions of $\alpha \in (0, 1)$, for $x_0 = 0.8$ and $b(x) \equiv 0$.

To complete Figure 6 we now explicitly find $\alpha_0(0.8)$ in the case $b \equiv 0$ under consideration, which can be used in Theorem 4.1 and Proposition 4.5 to prove global existence and uniqueness according to Theorem 5.2. Precisely, we prove that $\alpha_0(0.8)$ can be taken as 0.104.

In fact we carry out the calculations for a general $x_0 \in (0,1)$. As in the previous subsection, it suffices to find an $\alpha_0(x_0) > 0$ such that in a neighbourhood of 1

$$\frac{d}{dx}\mathbb{P}(X_t \in dx) < \frac{1}{\alpha}, \tag{6.19}$$

for all $t \in [0,T]$, whenever $(X_t)_{0 \leqslant t \leqslant T}$ is a solution to (2.1) on some $[0,T]$, $T > 0$, with $b \equiv 0$ and $\alpha \in (0, \alpha_0(x_0))$, i.e.

$$X_t = x_0 + \alpha e(t) + W_t - M_t, \qquad e(t) = \mathbb{E}(M_t), \qquad t \in [0,T].$$

Suppose that $\alpha \leqslant 1 - \eta$ for $\eta \in (0,1)$ with η to be chosen later and $x_0 < 1 - \alpha$. Then consider $x \in [x_0/(1-\alpha) + \varepsilon, 1]$ with $\varepsilon = (1/2)\,[1 - x_0/(1-\alpha)]$. The following simple Gaussian estimate will be our starting point:

$$\begin{aligned}\frac{d}{dx}\mathbb{P}(X_t \in dx) &= \frac{d}{dx}\mathbb{P}(Z_t - M_t \in dx) = \sum_{k=0}^{\infty}\frac{d}{dx}\mathbb{P}(Z_t - k \in dx, M_t = k)\\ &\leqslant \frac{1}{\sqrt{2\pi t}}\sum_{k=0}^{\infty}\exp\left(-\frac{(x + k - x_0 - \alpha e(t))^2}{2t}\right),\end{aligned} \tag{6.20}$$

where as usual $Z_t = X_t + M_t$, $(Z_t = x_0 + \alpha e(t) + W_t)_{0 \leqslant t \leqslant T}$ thus being a drifted Brownian motion.

We will need an *a priori* estimate of $e(t)$, similar to the one given in Proposition 2.2. However, in this simple case when $b \equiv 0$ we can do better. Indeed, using the fact that $M_t = \lfloor \sup_{s \leqslant t}(Z_s)_+ \rfloor$ (see (4.3)) and following (2.10)–(2.11), we see that

$$\begin{aligned}e(t) = \mathbb{E}(M_t) &= \mathbb{E}\left(\lfloor \sup_{s \leqslant t}(Z_s)_+ \rfloor\right)\\ &\leqslant x_0 + \alpha e(t) + \mathbb{E}\left(\sup_{s \leqslant t} W_s\right) \leqslant x_0 + \alpha e(t) + \left[\frac{2t}{\pi}\right]^{1/2},\end{aligned} \tag{6.21}$$

the last inequality following from (6.4). This yields

$$e(t) \leqslant \frac{1}{1-\alpha}(x_0 + C\sqrt{t}) \tag{6.22}$$

where $C = \sqrt{2/\pi}$. As in the previous section we look at what happens first in small time t.

Small t: By (6.22), we have that, for any $x \in [x_0/(1-\alpha) + \varepsilon, 1]$,

$$\begin{aligned}x - x_0 - \alpha e(t) &\geqslant x - \frac{x_0}{1-\alpha} - \frac{C\alpha}{1-\alpha}\sqrt{t} \geqslant \varepsilon - \frac{C\alpha}{1-\alpha}\sqrt{t}\\ &\geqslant \frac{\varepsilon}{2}\end{aligned} \tag{6.23}$$

when $\sqrt{t} \leqslant \varepsilon(1-\alpha)/(2C\alpha)$. Let $t_0 = [\varepsilon\eta/(2C(1-\eta))]^2$. Then since $\alpha \leqslant 1 - \eta$ we have that

$$t \leqslant t_0 \Rightarrow \sqrt{t} \leqslant \frac{\varepsilon\eta}{2C(1-\eta)} \leqslant \frac{\varepsilon(1-\alpha)}{2C\alpha}$$

so that (6.23) holds for all $t \leqslant t_0$. At this point we define $\beta := C(1-\eta)/\eta$, so that $\sqrt{t_0} = \varepsilon/(2\beta)$. From (6.23), we thus have that, for all $t \in [0, t_0]$,

$$\exp\left(-\frac{(x - x_0 - \alpha e(t))^2}{2t}\right) \leqslant \exp\left(-\frac{\varepsilon^2}{8t}\right)$$

and

$$\exp\left(-\frac{(x + k - x_0 - \alpha e(t))^2}{2t}\right) \leqslant \exp\left(-\frac{k^2}{2t}\right)$$

for $k \geqslant 1$. Hence by (6.20)

$$\frac{d}{dx}\mathbb{P}(X_t \in dx) \leqslant \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{\varepsilon^2}{8t}\right) + \frac{1}{\sqrt{2\pi t}} \sum_{k=1}^{\infty} \exp\left(-\frac{k^2}{2t}\right), \qquad t \in [0, t_0].$$

Note that

$$\frac{1}{\sqrt{2\pi t}} \sum_{k=1}^{\infty} \exp\left(-\frac{k^2}{2t}\right) \leqslant \frac{1}{\sqrt{2\pi t}} \sum_{k=1}^{\infty} \int_{k-1}^{k} \exp\left(-\frac{x^2}{2t}\right) dx = \frac{1}{2}.$$

Moreover

$$\sup_{t \,\geqslant\, 0} \left[\frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{\varepsilon^2}{8t}\right)\right] = \sqrt{\frac{2}{\pi}} \exp\left(-\frac{1}{2}\right) \frac{1}{\varepsilon},$$

and the supremum is attained when $t = \varepsilon^2/4$.

We thus arrive at

$$\frac{d}{dx}\mathbb{P}(X_t \in dx) \leqslant \sqrt{\frac{2}{\pi}} \exp\left(-\frac{1}{2}\right) \frac{1}{\varepsilon} + \frac{1}{2}, \qquad t \in [0, t_0]. \tag{6.24}$$

Large t (i.e. $t \geqslant t_0$): For fixed $x \in [x_0/(1-\alpha) + \varepsilon, 1]$ and $t > 0$ define

$$N(t, x) = \big(\lfloor \alpha e(t) + x_0 - x \rfloor\big)_+.$$

Then, by (6.22) again, we have that

$$\begin{aligned} N(t, x) &\leqslant \big(\alpha e(t) + x_0 - x\big)_+ \leqslant \left(\frac{C\alpha}{1-\alpha}\sqrt{t} + \frac{1}{1-\alpha}x_0 - x\right)_+ \\ &\leqslant \left(-\varepsilon + \frac{C\alpha}{1-\alpha}\sqrt{t}\right)_+ \leqslant \frac{C\alpha}{1-\alpha}\sqrt{t} \end{aligned} \tag{6.25}$$

for all $t > 0$. Moreover, for $t \geqslant t_0$,

$$
\begin{aligned}
&\frac{1}{\sqrt{2\pi t}} \sum_{k=0}^{\infty} \exp\left(-\frac{(x+k-x_0-\alpha e(t))^2}{2t}\right) \\
&\leqslant \frac{1}{\sqrt{2\pi t}} \sum_{\substack{k \geqslant 0 \\ k: x+k-x_0-\alpha e(t) \leqslant 1}} \exp\left(-\frac{(x+k-x_0-\alpha e(t))^2}{2t}\right) + \frac{1}{\sqrt{2\pi t}} \sum_{k \geqslant 1}^{\infty} \exp\left(-\frac{k^2}{2t}\right) \\
&\leqslant \frac{1}{\sqrt{2\pi t}} \sharp\big\{k \in \mathbb{N} \cup \{0\} : k \leqslant 1 + x_0 + \alpha e(t) - x\big\} + \frac{1}{2} \\
&\leqslant \frac{1}{\sqrt{2\pi t}} \left[N(t,x) + 2\right] + \frac{1}{2} \\
&\leqslant \frac{2}{\sqrt{2\pi t_0}} + \frac{C\alpha}{(1-\alpha)\sqrt{2\pi}} + \frac{1}{2} \leqslant 2\sqrt{\frac{2}{\pi}} \beta \frac{1}{\varepsilon} + \frac{\beta}{\sqrt{2\pi}} + \frac{1}{2},
\end{aligned} \tag{6.26}
$$

using the symbol $\sharp$ for the cardinal in the third line, and the fact that $\sqrt{t_0} = \varepsilon/(2\beta)$ and $\beta = C(1-\eta)/\eta \geqslant C\alpha/(1-\alpha)$.

We now consider two cases.

Case 1: when the RHS of (6.24) is bigger than the RHS of (6.26). This occurs if

$$
\begin{aligned}
2\sqrt{\frac{2}{\pi}} \beta \frac{1}{\varepsilon} + \frac{\beta}{\sqrt{2\pi}} \leqslant \sqrt{\frac{2}{\pi}} \exp\left(-\frac{1}{2}\right) \frac{1}{\varepsilon} \Leftrightarrow \quad & \frac{1}{2}(4+\varepsilon)\beta \leqslant \exp\left(-\frac{1}{2}\right) \\
\Leftrightarrow \quad & 1-\eta \leqslant \frac{2}{2 + C\exp(1/2)(4+\varepsilon)}.
\end{aligned}
$$

We therefore fix η such that $1-\eta = 2/(2 + C\exp(1/2)(4+\varepsilon))$. Thus we have a set of constraints on α in order to achieve (6.19), which are (recall (6.24))

$$
\begin{aligned}
\alpha &< \left(\sqrt{\frac{2}{\pi}} \exp(-\frac{1}{2}) \frac{1}{\varepsilon} + \frac{1}{2}\right)^{-1} \\
\alpha &\leqslant 1 - \eta = \frac{2}{2 + C\exp(1/2)(4+\varepsilon)},
\end{aligned}
$$

subject to $\alpha < 1 - x_0$. Substituting in the value of ε, the constraints become

$$
\alpha < \left(\frac{2C\exp(-1/2)(1-\alpha)}{1-\alpha-x_0} + \frac{1}{2}\right)^{-1} = f_1^{x_0}(\alpha) \tag{6.27}
$$

$$
\alpha \leqslant \frac{2}{2 + \left(4 + [1-\alpha-x_0]/[2(1-\alpha)]\right) C\exp(1/2)} = f_2^{x_0}(\alpha), \tag{6.28}
$$

still subject to $\alpha < 1 - x_0$, with $C = \sqrt{2/\pi}$. The first thing to note is that $f_1^{x_0}(\alpha)$ decreases to 0 on the interval $\alpha \in [0, 1-x_0)$. By plotting the curves of $f_1^{x_0}(\alpha)$ and $f_2^{x_0}(\alpha)$ ($\alpha \in [0, 1-x_0)$) for different values of x_0, we see that (6.27) is more stringent than (6.28) for large values of x_0, while the reverse holds for small values of x_0. Indeed the critical value of x_0 is approximately 0.51.

Hence for $x_0 \gtrapprox 0.51$, if (6.27) is satisfied subject to $\alpha < 1 - x_0$, then we achieve (6.19). Thus for $x_0 \gtrapprox 0.51$, we choose $\alpha_0(x_0) = \min[1 - x_0, \widetilde{\alpha}_0(x_0)]$ where $\widetilde{\alpha}_0(x_0)$ is such that

$$1 = \widetilde{\alpha}_0(x_0)\left(\frac{2C\exp(-1/2)(1 - \widetilde{\alpha}_0(x_0))}{1 - \widetilde{\alpha}_0(x_0) - x_0} + \frac{1}{2}\right),$$

which solves as a second-order equation. For instance, this yields $\alpha_0(0.8) \approx 0.104$, which completes Figure 6 above.

Now, if $x_0 \lessapprox 0.51$ then (6.28) is the more stringent. In this case we must solve $\widetilde{\alpha}_0(x_0) = f_2^{x_0}(\widetilde{\alpha}_0(x_0))$ and then choose $\alpha_0(x_0) = \min[1 - x_0, \widetilde{\alpha}_0(x_0)]$. For example, $\alpha_0(x_0) \approx 0.264$ when $x_0 = 0.4$. We note here, however, that as $x_0 \downarrow 0$, $f_2^{x_0}(\alpha) \to 4/[4 + 9C\exp(1/2)] \approx 0.2525$ for all $\alpha \in (0, 1)$. We can in fact hope to do better when x_0 is small.

Case 2: when the RHS of (6.26) is bigger than the RHS of (6.24). In order to try and do better for x_0 small, we now look at this case, which occurs when $(4 + \varepsilon)\beta \geqslant 2\exp(-1/2)$: here ε depends upon x_0 but β does not, so that β acts as a free parameter that can be tuned accordingly to the value of x_0. Such a strategy turns out to be efficient when x_0 is small. Indeed, our constraints now become

$$\alpha < \left(\sqrt{\frac{2}{\pi}}\frac{2\beta}{\varepsilon} + \frac{\beta}{\sqrt{2\pi}} + \frac{1}{2}\right)^{-1} = \left(\left(\frac{4}{\varepsilon} + 1\right)\frac{\beta}{\sqrt{2\pi}} + \frac{1}{2}\right)^{-1} \tag{6.29}$$

$$\alpha \leqslant 1 - \eta = 1 - \frac{C}{\beta + C} = \frac{\beta}{\beta + C}, \tag{6.30}$$

again subject to $\alpha < 1 - x_0$, where we may vary β, as long as $(4+\varepsilon)\beta \geqslant 2\exp(-1/2)$. Now, optimizing (6.29) and (6.30) over β by equalizing the two right-hand sides and thus by solving

$$\beta\left(\left(\frac{4}{\varepsilon} + 1\right)\frac{C\beta}{2} + \frac{1}{2}\right) = \beta + C \Leftrightarrow \left(\frac{4}{\varepsilon} + 1\right)C\beta^2 - \beta - 2C = 0,$$

we would like to take β such that

$$2\left(\frac{4}{\varepsilon} + 1\right)C\beta = 1 + \sqrt{1 + 4\left(\frac{4}{\varepsilon} + 1\right)2C^2}. \tag{6.31}$$

This can be done provided it remains true that $(4 + \varepsilon)\beta \geqslant 2\exp(-1/2)$, i.e. that

$$(4 + \varepsilon)\left(1 + \sqrt{1 + 4\left(\frac{4}{\varepsilon} + 1\right)2C^2}\right) \geqslant 4C\exp\left(-\frac{1}{2}\right)\left(\frac{4}{\varepsilon} + 1\right),$$

which we can see numerically holds for $\varepsilon > 0.15$ (which basically says that x_0 must be small, as announced).

If we suppose that $\alpha \leqslant 0.42$ (we will see below that we do not in fact lose anything by doing this), then since $\varepsilon = (1 - x_0 - \alpha)/2(1 - \alpha)$, it is certainly true that $\varepsilon > 0.15$ when x_0 is small. Indeed it is true when $x_0 < 0.28$. In this case,

$\alpha \leqslant 0.42 < 1 - 0.28 < 1 - x_0$. By (6.29) and (6.31), we then take $\alpha_0(x_0)$ such that

$$\begin{aligned}\alpha_0(x_0) &= \left(\left(\frac{4}{\varepsilon} + 1 \right) \frac{C\beta}{2} + \frac{1}{2} \right)^{-1} \\ &= 4 \left(3 + \sqrt{1 + 4 \left(\frac{4}{\varepsilon} + 1 \right) 2C^2} \right)^{-1} \\ &= 4 \left(3 + \sqrt{1 + 4 \left(\frac{8(1-\alpha_0(x_0))}{1 - \alpha_0(x_0) - x_0} + 1 \right) 2C^2} \right)^{-1}.\end{aligned}$$

Then it follows that whenever $x_0 < 0.28$, (6.19) holds for all $\alpha < \alpha_0(x_0)$. Taking $x_0 = 0$ we calculate that

$$\alpha_0(0) = 4 \left(3 + \sqrt{1 + 72C^2} \right)^{-1} \approx 0.41,$$

which is much better than that achieved above for small x_0. This must be also compared with the value obtained for $\alpha_0(0)$ in the Ornstein-Uhlenbeck setting when $\lambda = 0$: referring to Figure 3, it is approximately equal to 0.37, which is a bit less but not so different. The difference between both is not so surprising. Indeed the method used in the Brownian setting is more straightforward since it benefits from the Gaussian property of Z_t.

7. Appendix: Proof of Lemma 3.3

First Step. We first discuss the solvability of (3.4). We start with the following case: we assume that $\chi_0 = x_0 \in (-\infty, 1)$, and that b and e are smooth and bounded, with bounded derivatives of any order. Then, by [9, Th 1.10, Chap. VI], we know that the generator of χ, namely the family of second-order differential operators

$$\left(\mathcal{L}_{s,x} := \big(b(\cdot) + \alpha e'(s) \big) \partial_x + \frac{1}{2} \partial^2_{xx} \right)_{0 \leqslant s \leqslant T},$$

admits a Green function $G : [0,T]^2 \times (-\infty, 1]^2 \ni (s,t,x,y) \mapsto G(s,x,t,y)$. For a given $(t,y) \in [0,T] \times (-\infty, 1]$, the function $[0,t) \times (-\infty, 1] \ni (s,x) \mapsto G(s,x,t,y)$ is a classical solution of the PDE

$$\partial_s G(s,x,t,y) + \mathcal{L}_{s,x} G(s,x,t,y) = 0,$$

with $G(s,1,t,y) = 0$, for $s \in [0,t)$ and $G(s,x,t,y) \to \delta_0(x-y)$ as $s \nearrow t$, where δ_0 is the Dirac mass at point 0 (pay attention that our definition of the Green function obeys the convention used in probability theory: it is thus reversed in time in comparison with the standard notation used in the PDE literature). Following [8, Th. 5, Sec. 5, Chap. 9], for a given $(s,x) \in [0,T) \times (-\infty, 1)$, the function $(s,T] \times (-\infty, 1] \ni (t,y) \mapsto G(s,x,t,y)$ is also known to be the Green function of the adjoint operator

$$\partial_t \cdot + \partial_y \big[\big(b(y) + \alpha e'(t) \big) \cdot \big] - \frac{1}{2} \partial^2_{yy} \cdot,$$

with a Dirichlet boundary condition on $[0,T]\times\{1\}$. In particular, $G(s,x,t,1)=0$ and $G(s,x,t,y)\to\delta_0(y-x)$ as $t\searrow s$. When $\chi_0=x_0$, we then set

$$p(t,y)=G(0,x_0,t,y),\quad t\in(0,T],\ y\in(-\infty,1]. \tag{7.1}$$

By [9, Th. 1.10, Chap. VI] (applied to the adjoint operator), we know that $p(t,y)$ (as the solution to (3.4) under the current smoothness assumptions with $X_0=x_0$) decays exponentially fast as t tends to 0 and y stays away from x_0. This proves that p is continuous on any compact subset of $([0,T]\times(-\infty,1])\setminus\{(0,x_0)\}$.

Second Step. Still in the smooth framework, we now make the connection with the diffusion process χ when $X_0=x_0$. Precisely, the point is to prove that the definition of p in (3.3) is coherent with that in (7.1). Put it differently, we must check that the right-hand side of (3.3) coincides with the definition of p in (7.1) when $\chi_0=x_0$. Given a smooth function $\phi:[0,T]\times(-\infty,1]\to\mathbb{R}$, with a compact support, the analysis of the Green function in [9, Th. 1.10, Chap. VI] says that the PDE

$$\partial_s u(s,x)+\big(b(x)+\alpha e'(s)\big)\partial_x u(s,x)+\frac{1}{2}\partial^2_{xx}u(s,x)+\phi(s,x)=0,$$

for $(s,x)\in(0,T]\times(-\infty,1)$, with $u(T,x)=0$, $x\in(-\infty,1)$, as initial condition and $u(s,1)=0$, $s\in[0,T]$, as Dirichlet boundary condition, admits a (unique) classical solution

$$u(s,x)=\int_s^T\int_{-\infty}^1 G(s,x,t,y)\phi(t,y)dydt,\quad s\in[0,T),\ x\leqslant 1. \tag{7.2}$$

Moreover, u is bounded and continuous on $[0,T]\times(-\infty,1]$ and is once continuously differentiable in time and twice differentiable in space on $[0,T)\times(-\infty,1)$. Therefore, we can expand $(u(t\wedge\tau_1,\chi_{t\wedge\tau_1}))_{0\leqslant t\leqslant T}$ by Itô's formula, with $\tau_1=\inf\{t\geqslant 0:\chi_t\geqslant 1\}$. We then have the well-known representation formula:

$$u(0,x_0)=\mathbb{E}\int_0^{T\wedge\tau_1}\phi(t,\chi_t)dt. \tag{7.3}$$

By equalizing (7.2) and (7.3), we deduce that

$$\mathbb{E}\int_0^{T\wedge\tau_1}\phi(t,\chi_t)dt=\int_0^T\int_{-\infty}^1 p(t,y)\phi(t,y)dydt. \tag{7.4}$$

Writing

$$\begin{aligned}\mathbb{E}\int_0^{T\wedge\tau_1}\phi(t,\chi_t)dt&=\int_0^T\mathbb{E}\big[\phi(t,\chi_t)\mathbf{1}_{\{t<\tau_1\}}\big]dt\\&=\int_0^T\int_{-\infty}^1\phi(t,y)\mathbb{P}\big(\chi_t\in dy,t<\tau_1\big)dt,\end{aligned}$$

we deduce that

$$\int_0^T\int_{-\infty}^1\phi(t,y)\mathbb{P}\big(\chi_t\in dy,t<\tau_1\big)dt=\int_0^T\int_{-\infty}^1 p(t,y)\phi(t,y)dydt, \tag{7.5}$$

so that (3.3) holds in the smooth setting.

In the same framework, we then prove (3.5). The cumulative distribution function of τ_1 is given by

$$\mathbb{P}\big(\tau_1 \leqslant t\big) = 1 - \mathbb{P}\big(\tau_1 > t\big) = 1 - \int_{-\infty}^{1} p(t,y)dy.$$

By [9, Th. 1.10, Chap. VI], we can differentiate the above expression with respect to t and exchange the derivative and the integral. From (3.4), we deduce:

$$\begin{aligned}\frac{d}{dt}\mathbb{P}\big(\tau_1 \leqslant t\big) &= -\int_{-\infty}^{1} \partial_t p(t,y)dy \\ &= \int_{-\infty}^{1} \partial_y\big([\alpha e'(t) + b(y)]p(t,y)\big)dy - \frac{1}{2}\int_{-\infty}^{1} \partial^2_{yy}p(t,y)dy, \quad t \geqslant 0.\end{aligned}$$

Again by [9, Th. 1.10, Chap. VI], we know that both $p(t,y)$ and $\partial_y p(t,y)$ tend to 0 exponentially fast as $y \to -\infty$. So, using the boundary condition $p(t,1) = 0$, we obtain (3.5).

Third Step. We now aim at proving the same results, still with $\chi_0 = x_0$, but under the original assumptions on b and e. The strategy is to use a mollification argument. In order to do so, we must prove that, in the smooth setting, p and $\partial_y p$ can be bounded by constants that only depend upon T, A, K, Λ and x_0, where A is such that $\sup_{t \leqslant T} |e'(t)| \leqslant A$.

We thus go back to the case when b and e are smooth and bounded with bounded derivatives of any order. By Proposition 3.7, for $t \in [0,T]$ small enough, we already have a Gaussian bound for $\partial_y p(t,y)$ in terms of T, A, K and Λ only (notice that the argument applies since we know that p indeed satisfies PDE (3.4) in the smooth setting). With the same notation as in the statement of Proposition 3.7 (using in addition Proposition 3.5), the bound is of the form

$$|\partial_y p(t,y)| \leqslant \frac{C}{t}\exp\left(-\frac{|\xi_t^{x_0} - y|^2}{Ct}\right), \tag{7.6}$$

for $t \in (0,\delta]$, where $(\xi_t^{x_0})_{t \geqslant 0}$ is given by (3.14) and C and δ are constants that depend on T, A, K and Λ only. Plugging this bound into (3.10) and repeating the Gaussian convolution argument used in (3.31), we deduce that, for $t \in (0,\delta]$,

$$p(t,y) \leqslant \frac{C}{\sqrt{t}}\exp\left(-\frac{|\xi_t^{x_0} - y|^2}{Ct}\right), \tag{7.7}$$

up to a new value of C.

Actually, (7.6) and (7.7) can be seen as bounds for the Green function G and its derivative in small time since $p(t,y) = G(0,x_0,t,y)$. In the same way, we could prove similar bounds for $G(s,x,t,y)$ and $\partial_y G(s,x,t,y)$ when $t-s \in (0,\delta]$. By a standard chaining argument, we then deduce that (7.6) and (7.7) are valid on the whole $[0,T]$, for a possibly new value of C. Indeed, for a given $t \in (0,T]$, we can consider a sequence $0 = t_0 < t_1 < \cdots < t_N = t$ such that $t_{i+1} - t_i \leqslant \delta$. Then, by

using the Markov structure of χ, we have:

$$p(t,y) = \int_0^t \int_{(-\infty,1]^{N-1}} \prod_{i=1}^N G(t_{i-1},t_i,z_{i-1},z_i) dz_1 \dots dz_{N-1},$$

with the convention $z_0 = 0$ and $z_N = y$. Noticing that N can be assumed to be bounded from above and using again a Gaussian convolution argument, we deduce that (7.6) and (7.7) can be extended to the whole $(0,T]$.

Fourth Step. We still assume that the coefficients are smooth and bounded, with bounded derivatives of any order. By the third step, we are then able to reduce the PDE (3.4) to an heat PDE with a non-trivial source term:

$$\partial_t p(t,y) - \frac{1}{2}\partial^2_{yy} p(t,y) = -\partial_y\big[\big(b(y)+\alpha e'(t)\big)p(t,y)\big], \quad t \in (0,T],\ y<1.$$

For any compact subset $\mathcal{K} \subset (0,T] \times (-\infty,1]$, we can consider a smooth cut-off function $\eta : [0,T] \times \mathbb{R} \to \mathbb{R}_+$ matching 1 on $\mathcal{K}$ and vanishing outside another compact subset $\mathcal{K}' \subset (0,T] \times \mathbb{R}$, $\mathcal{K} \subset \mathcal{K}'$. Then, the function $(0,T] \times (-\infty,1] \ni (t,y) \mapsto [\eta p](t,y)$ satisfies the heat equation:

$$\big(\partial_t - \frac{1}{2}\partial^2_{yy}\big)\big[\eta p\big](t,y) = f(t,y), \tag{7.8}$$

with $[\eta p](0,\cdot) = 0$ as initial condition and with $[\eta p](t,1) = 0$ and $[\eta p](t,y) = 0$ for $t \in [0,T]$ and $|y|$ large enough as boundary conditions, where f is a smooth function which can be bounded in terms of T, A, η, K and Λ only. By [8, Th. 4, Sec. 2, Chap. 7], we deduce that ηp and $\partial_y[\eta p]$ are $(1/4,1/2)$-Hölder continuous in (t,x), the Hölder constant depending on T, A, η, K and Λ only. Therefore, on any compact subset $\mathcal{K}$ of $(0,T]\times(-\infty,1]$, p and $\partial_y p$ are $(1/4,1/2)$-Hölder continuous in (t,x), the Hölder constant depending on T, A, $\mathcal{K}$, K and Λ only. By the same argument, on any compact subset $\mathcal{K}$ of $[0,T]\times\mathcal{I}$, $\mathcal{I}$ being a finite interval included in $(-\infty,1]$ not containing x_0, p and $\partial_y p$ are also $(1/4,1/2)$-Hölder continuous in (t,x), the Hölder constant depending on T, A, $\mathcal{K}$, K and Λ only. Indeed, in such a case, $p(t,y)$ and $\partial_y p(t,y)$ tend to 0 as t tends to 0 and y stays in $\mathcal{I}$, so that the compact support $\mathcal{K}'$ of η can be assumed to be included in $[0,T]\times\mathbb{R}$ (and not necessarily in $(0,T]\times\mathbb{R}$). In the end, we deduce that, on any compact subset $\mathcal{K}$ of $([0,T]\times(-\infty,1])\setminus\{(0,x_0)\}$, p and $\partial_y p$ are $(1/4,1/2)$-Hölder continuous in (t,x), the Hölder constant depending on T, A, $\mathcal{K}$, K and Λ only.

In order to tackle the second-order derivatives in space, we assume that $\mathcal{K}' \subset (0,T]\times(-\infty,1)$, $\mathcal{K}'$ being the support of η. Then, the function $(0,T]\times\mathbb{R} \ni (t,y) \mapsto [\eta p](t,y)$ (with $[\eta p](t,y) = 0$ for $y > 1$) satisfies (7.8) on the whole $(0,T]\times\mathbb{R}$, so that it can be represented as a standard Gaussian convolution. Then, by Calderon and Zygmund estimates, see [19, Eq. (0.4), App. A], for any $\varsigma \geqslant 1$, the $L^\varsigma([0,T]\times\mathbb{R}, dt\otimes dy)$-norms of $\partial_t[\eta p]$ and $\partial_{yy}[\eta p]$ are bounded in terms of A, η, K, Λ and T only. Therefore, on any compact subset $\mathcal{K}$ of $(0,T]\times(-\infty,1)$, for any $\varsigma \geqslant 1$, the $L^\varsigma(\mathcal{K}, dt\otimes dy)$-norms of $\partial_t p$ and $\partial_{yy} p$ are bounded in terms of T, A, $\mathcal{K}$, K and Λ only. For example, when $\mathcal{K}$ is a cylinder of the form $[\delta,T]\times[y-1,y+1]$, (7.6) and

(7.7) say that

$$\int_{\mathcal{K}} \big(|\partial_t p(t,z)|^{\varsigma} + |\partial^2_{yy} p(t,z)|^{\varsigma} \big) dt dz \leqslant C_{\varsigma,\delta} \int_{\delta}^{T} \exp\left(-\frac{|\xi^{x_0}_s - y|^2}{C_{\varsigma,\delta}} \right) ds, \tag{7.9}$$

for a constant $C_{\varsigma,\delta}$ that is independent of x_0 and y.

Fifth Step. We now have all the required ingredients to go back to the original framework. The point is to approximate b and e by two sequences $(b^n)_{n\geqslant 1}$ and $(e^n)_{n\geqslant 1}$ (for the topology of uniform convergence on compact sets) that satisfy the previous smoothness conditions. The associated solutions to the PDE (3.4) are denoted by $(p^n)_{n\geqslant 1}$. On any compact subset $\mathcal{K} \subset ([0,T]\times(-\infty,1]) \setminus \{(0,x_0)\}$, the sequences $(p^n)_{n\geqslant 1}$ and $(\partial_y p^n)_{n\geqslant 1}$ are uniformly bounded and $(1/4,1/2)$-Hölder continuous in (t,x). Therefore, there exists a subsequence $(\varphi(n))_{n\geqslant 1}$ such that $(p^{\varphi(n)})_{n\geqslant 1}$ and $(\partial_y[p^{\varphi(n)}])_{n\geqslant 1}$ are uniformly convergent on compact subsets of $([0,T]\times(-\infty,1]) \setminus \{(0,x_0)\}$. Similarly, we can assume that the sequences $(\partial_t[p^{\varphi(n)}])_{n\geqslant 1}$ and $(\partial^2_{yy}[p^{\varphi(n)}])_{n\geqslant 1}$ are weakly convergent for the $L^{\varsigma}(\mathcal{K}, dt\otimes dy)$ topology, for any $\varsigma \geqslant 1$ and any compact subset $\mathcal{K} \subset (0,T]\times(-\infty,1)$. The limit function of the sequence $(p^{\varphi(n)})_{n\geqslant 1}$ is denoted by p: clearly, it is a solution of (3.4) (in the Sobolev sense), with $p(t,1) = 0$ for $t > 0$, as Dirichlet boundary condition. By (7.7), it tends to 0 as t tends to 0 and y stays away from x_0. Moreover, $\partial_y p$ exists and is continuous on $([0,T]\times(-\infty,1]) \setminus \{(0,x_0)\}$.

Then, we can prove (3.3). Indeed, we know that $(p^{\varphi(n)})_{n\geqslant 1}$ converges toward p uniformly on compact subsets of $([0,T]\times(-\infty,1]) \setminus \{(0,x_0)\}$. Moreover, it is standard to prove that the solution χ^n to the SDE (3.2), but driven by b^n and e^n instead of b and e, converges in law (as $n \to \infty$) toward χ on the space $\mathcal{C}([0,T],\mathbb{R})$ of continuous functions on $[0,T]$ endowed with the uniform topology. Denoting by $\tau^n_1 := \inf\{t \geqslant 0 : \chi^n_t \geqslant 1\}$, we claim that the pair $(\tau^n_1 \wedge T, \chi^n)$ converges in law toward $(\tau_1 \wedge T, \chi)$ on the product space $\mathbb{R}\times\mathcal{C}([0,T],\mathbb{R})$ endowed with the product topology. Indeed, the mapping

$$\mathcal{C}([0,T],\mathbb{R}) \ni x \mapsto (\inf\{t \geqslant 0 : x(t) \geqslant 1\}) \wedge T$$

is continuous at any path x satisfying the “crossing property”:

$$\big(\exists t \in [0,T) : x(t) \geqslant 1\big) \Rightarrow \big(\forall p \geqslant 1,\ \exists t_p \in (t, t+1/p) : x(t_p) > 1\big), \tag{7.10}$$

and (7.10) holds for a.e. trajectory of χ because of the Brownian part in the dynamics of χ, see (3.2). Recalling that (7.4) holds true in the smooth setting, that is with p and τ_1 therein replaced by p^n and τ^n_1, $n \geqslant 1$, we can pass to the limit along the subsequence $(\varphi(n))_{n\geqslant 1}$ (to pass to the limit in the right-hand side of (7.4), use the boundedness of the support of ϕ and the uniform convergence of $(p^{(\varphi(n))})_{n\geqslant 1}$ toward p on compact subsets of $(0,T]\times(-\infty,1]$). We deduce that (7.4) holds in the general setting as well, from which we deduce that (7.5) also holds in the general setting, provided $\chi_0 = x_0$.

By the same approximation argument, we can prove that (3.5) holds in the general setting as well, provided $\chi_0 = x_0$.

Below, we shall denote p^{x_0} for p to indicate the dependence upon x_0. By (3.3), p^{x_0} depends upon x_0 in a measurable way.

Sixth Step. To complete the proof, it remains to discuss what happens when χ_0 does not reduce to a Dirac mass. The point is then prove that

$$p(t,y) = \int_{-\infty}^{1} p^x(t,y)\mathbb{P}(\chi_0 \in dx), \quad t \geqslant 0, \ y \leqslant 1, \tag{7.11}$$

is the right candidate for solving the Fokker-Planck equation and for making the connection with χ.

We first check that the definition (7.11) makes sense when $t > 0$. By applying the Markov property for χ, this will directly prove (3.3). From (7.7), we know that $p^x(t,y) \leqslant (C/t^{1/2})\exp[-|\xi_t^x - y|^2/(Ct)]$. From (3.19), we can find some $C' > 0$ such that

$$\begin{aligned}
&\int_{-\infty}^{1} \exp\left(-\frac{|\xi_t^x - y|^2}{Ct}\right)\mathbb{P}\big(\chi_0 \in dx\big) \\
&\leqslant \int_{-\infty}^{1} \exp\left(-\frac{|x - \xi_{-t}^y|^2}{C't}\right)\mathbb{P}\big(\chi_0 \in dx\big) \\
&= \exp\left(-\frac{|1 - \xi_{-t}^y|^2}{C't}\right) + \int_{-\infty}^{1} \frac{2(x - \xi_{-t}^y)}{C't}\exp\left(-\frac{|x - \xi_{-t}^y|^2}{C't}\right)\mathbb{P}\big(\chi_0 < x\big)dx \\
&\leqslant \exp\left(-\frac{|1 - \xi_{-t}^y|^2}{C't}\right) + \int_{-\infty}^{1} \frac{2|x - \xi_{-t}^y|\,\mathbb{E}(|2 - \chi_0|)}{C't(2-x)}\exp\left(-\frac{|x - \xi_{-t}^y|^2}{C't}\right)dx,
\end{aligned} \tag{7.12}$$

which is finite since $\mathbb{E}|\chi_0| < \infty$, the third line following from an integration by parts and the last one from Markov's inequality. This also proves that $p(t,y) \to 0$ when $y \to -\infty$.

The bound (7.12) can be easily modified into a domination argument for $p^x(t,y)$, for (t,y) in compact subsets of $(0,T] \times (-\infty,1]$. By the Lebesgue Dominated Convergence Theorem, this proves that p is continuous on $(0,T] \times (-\infty,1]$. When $\chi_0 \leqslant 1-\epsilon$, the same domination argument shows that continuity holds on any compact subset of $([0,T] \times (-\infty,1]) \setminus (\{0\} \times (-\infty,1-\epsilon])$, since the integrals in (7.12) run in that case from $-\infty$ to $1-\epsilon$.

Using (7.6), we can prove in a similar way that p is continuously differentiable in y on $(0,T] \times (-\infty,1]$, with

$$\partial_y p(t,y) = \int_{-\infty}^{1} \partial_y p^x(t,y)\mathbb{P}(\chi_0 \in dx), \quad t > 0, \ y \leqslant 1, \tag{7.13}$$

and that $\partial_y p(t,y) \to 0$ when $y \to -\infty$ When $\chi_0 \leqslant 1-\epsilon$, continuous differentiability holds on any compact subset of $([0,T] \times (-\infty,1]) \setminus (\{0\} \times (-\infty,1-\epsilon])$.

By combining the same domination argument with (7.9), we can also prove that p admits Sobolev derivatives of order 1 in t and of order 2 in y in any L^ς, $\varsigma \geqslant 1$, on any compact subsets of $(0,T] \times (-\infty,1)$, the derivatives being given as the integrals of the derivatives of p^x with respect to the law of χ_0. It is then plain to check that p satisfies (3.4).

The last point is thus to check that p satisfies (3.5). By Markov's property,

$$\begin{aligned}\mathbb{P}(\tau_1 \leqslant t) &= \int_{-\infty}^{1} \mathbb{P}(\tau_1 \leqslant t | \chi_0 = x)\,\mathbb{P}(\chi_0 \in dx) \\ &= -\frac{1}{2}\int_{-\infty}^{1}\left[\int_0^t \partial_y p^x(s,1)ds\right]\mathbb{P}(\chi_0 \in dx).\end{aligned} \tag{7.14}$$

In order to prove (3.5), we must exchange the two integrals. To exchange the two integrals, we specify the upper bound for $|\partial_y p^x(t,1)|$. Using (7.6) and (3.19) and recalling that $\xi^1_\cdot \equiv 1$, we deduce that $|\partial_y p^x(t,1)| \leqslant (C/t)\exp[-|1-x|^2/(Ct)]$, from which it holds that

$$\begin{aligned}&\int_{-\infty}^{1}\int_0^t |\partial_y p^x(s,1)| ds\,\mathbb{P}(\chi_0 \in dx) \\ &\leqslant \int_{-\infty}^{1-\epsilon}\int_0^t \frac{C}{s}\exp\left(-\frac{|1-x|^2}{Cs}\right) ds\,\mathbb{P}(\chi_0 \in dx) \\ &\quad + \int_{1-\epsilon}^{1}\int_0^t \frac{C}{s}\exp\left(-\frac{|1-x|^2}{Cs}\right) ds\,\mathbb{P}(\chi_0 \in dx).\end{aligned} \tag{7.15}$$

The first integral in the right-hand side is clearly bounded. To tackle the second one, we make use of the assumption $\mathbb{P}(\chi_0 \in dx) \leqslant \beta(1-x)dx$ for $x \in (1-\epsilon, 1]$. We obtain

$$\int_{1-\epsilon}^{1}\int_0^t \frac{C}{s}\exp\left(-\frac{|1-x|^2}{Cs}\right) ds\,\mathbb{P}(\chi_0 \in dx) \leqslant \beta\int_0^{\epsilon}\int_0^t \frac{Cx}{s}\exp\left(-\frac{|x|^2}{Cs}\right) ds\,dx < \infty. \tag{7.16}$$

Going back to (7.14), we obtain

$$\mathbb{P}(\tau_1 \leqslant t) = -\frac{1}{2}\int_0^t \left(\int_{-\infty}^{1} \partial_y p^x(s,1)\mathbb{P}(\chi_0 \in dx)\right) ds = -\frac{1}{2}\int_0^t \partial_y p(s,1) ds,$$

the last part following from (7.13). Since the mapping $(0,T] \ni t \mapsto \partial_y p(t,1)$ is continuous, we deduce that the mapping $(0,T] \ni t \mapsto \mathbb{P}(\tau_1 \leqslant t)$ is continuously differentiable. It then remains to check the continuous differentiability at time $t=0$. To this end, it is sufficient to prove that $\partial_y p(\cdot,1)$ is continuous at $t=0$. Following (7.15) and (7.16), it is clear that, for any $\epsilon' \in (0,\epsilon)$,

$$\lim_{t\searrow 0}\partial_y p(t,1) = \lim_{t\searrow 0}\int_{1-\epsilon'}^{1} \partial_y p^x(t,1)\mathbb{P}(\chi_0 \in dx) = \lim_{t\searrow 0}\int_{1-\epsilon'}^{1} \partial_y p^x(t,1)p_0(x)dx, \tag{7.17}$$

where $p_0(x) = (d/dx)\mathbb{P}(\chi_0 \in dx)$, which is assumed to make sense on $(1-\epsilon', 1)$. By Proposition 3.7, we know that, for some constant $C > 0$,

$$|\partial_y p^x(t,y) - \partial_y q(t,x,y)| \leqslant \frac{C}{\sqrt{t}}\exp\left(-\frac{|x-1|^2}{Ct}\right),$$

which proves, by using the bound $p_0(x) \leqslant \beta(1-x)$ and by letting ϵ' tend to 0 in (7.17), that

$$\lim_{t\searrow 0} \partial_y p(t,1) = \lim_{\epsilon'\searrow 0}\lim_{t\searrow 0} \int_{1-\epsilon'}^{1} \partial_y q(t,x,1)p_0(x)dx. \tag{7.18}$$

We then introduce the killed Gaussian kernel with reflection at point 1:

$$\hat{q}(t,x,y) = \frac{1}{\sqrt{2\pi t}}\left[\exp\left(-\frac{|x-y|^2}{2t}\right) - \exp\left(-\frac{|x-(2-y)|^2}{2t}\right)\right].$$

Now a formula like (3.10) for expressing $\partial_y p_e^x(t,y)$ in terms of $\partial_y q(t,x,y)$ can be proven for expressing $\partial_y q(t,x,y)$ in terms of $\partial_y \hat{q}(t,x,y)$ (replacing $\alpha e'(s)+b(1)$ by $b(z)-b(1)$ in (3.10)). Recalling $|b(z)-b(1)| \leqslant K|z-1|$, we obtain (for a possibly new value of C which is allowed to increase from line to line)

$$\begin{aligned}
&|\partial_y q(t,x,1) - \partial_y \hat{q}(t,x,1)| \\
&\leqslant C\int_0^t\int_{-\infty}^1 \frac{C(1-z)}{s(t-s)}\exp\left(-\frac{|\xi_s^x - z|^2}{Cs}\right)\exp\left(-\frac{|1-z|^2}{C(t-s)}\right)dzds \\
&\leqslant C\int_0^t\int_{-\infty}^1 \frac{C}{s(t-s)^{1/2}}\exp\left(-\frac{|\xi_s^x - z|^2}{Cs}\right)dzds \\
&\leqslant C.
\end{aligned}$$

Recalling (7.18), we deduce that:

$$\begin{aligned}
\lim_{t\searrow 0}\partial_y p(t,1) &= \lim_{\epsilon'\searrow 0}\lim_{t\searrow 0}\int_{1-\epsilon'}^1 \partial_y \hat{q}(t,x,1)p_0(x)dx, \\
&= -\lim_{\epsilon'\searrow 0}\lim_{t\searrow 0} 2t^{-1/2}\int_{1-\epsilon'}^1 g'\left(\frac{x-1}{t^{1/2}}\right)p_0(x)dx,
\end{aligned}$$

where g stands for the standard Gaussian kernel. Since p_0 is assumed to be differentiable at 1, with $p_0(1)=0$, we can write $p_0(x) = (x-1)p_0'(1) + o(x-1)$, where $o(\cdot)$ stands for the Landau notation. We deduce that the limit of $\partial_y p(t,1)$ as $t \searrow 0$ must be $p_0'(1)$. Following (7.15) and (7.16), we have an explicit bound for the supremum norm of $\partial_y p(\cdot,1)$ in terms of α, β, b, T and the supremum norms of e and e' only.

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François Delarue, Sylvain Rubenthaler, Laboratoire J.-A. Dieudonné, Université de Nice Sophia-Antipolis, Parc Valrose, 06108 Nice Cedex 02, France.
E-mail address: `delarue@unice.fr,rubentha@unice.fr`

James Inglis, Etienne Tanré, Equipe Tosca, INRIA Sophia-Antipolis Méditerranée, 2004 route des lucioles, BP 93, 06902 Sophia Antipolis Cedex, France
E-mail address: `James.Inglis@inria.fr,Etienne.Tanre@inria.fr`